Duration and Asset Prices

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Consider the cash flow stream $C_1, C_2, \cdots, C_T$. Let PV be the present value of this cash flow stream discounted at the rate r. Then

$$\begin{aligned} PV &= \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_T}{(1+r)^T} \\ &= \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} \end{aligned}$$

The **duration** of the cash flow is defined as

$$\begin{aligned} D &= \frac{1}{PV} \sum_{t=1}^{T} \frac{t \cdot C_t}{(1+r)^t} \\ &= \frac{1}{PV} \left(\frac{C_1}{1+r} + \frac{2C_2}{(1+r)^2} + \frac{3C_3}{(1+r)^3} + \cdots + \frac{TC_T}{(1+r)^T} \right) \end{aligned}$$

Interpretation. Since

$$\frac{1}{PV}\frac{C_s}{(1+r)^s} = \frac{C_s/(1+r)^s}{\sum_{t=1}^{T} C_t/(1+r)^t}$$

$\frac{1}{PV}\frac{C_s}{(1+r)^s}$ is the proportion of the present value of the cash flow stream that is attributable to the cash flow at date s. The duration of the cash flow is the **weighted average of the dates**, where the weights are given by these proportions. In other words,

$$D = w_1 \cdot 1 + w_2 \cdot 2 + \cdots + w_T \cdot T$$

where

$$w_s = \frac{C_s/(1+r)^s}{\sum_{t=1}^{T} C_t/(1+r)^t}$$

The duration of the cash flow stream is a measure of how far in the future the cash flow arrives Duration is also a first-order approximation of the proportional change in the present value arising from a change in its discount rate.

$$PV = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t}$$

$$\frac{d}{dr} PV = \sum_{t=1}^{T} \frac{(-t) \cdot C_t}{(1+r)^{t+1}} = -\frac{1}{1+r} \sum_{t=1}^{T} \frac{t \cdot C_t}{(1+r)^t}$$

So,

$$\begin{aligned} \frac{d\,(PV)\,/dr}{PV} &= -\frac{1}{1+r}\frac{1}{PV} \sum_{t=1}^{T} \frac{t \cdot C_t}{(1+r)^t} \\ &= -\frac{D}{1+r} \end{aligned}$$

For small r,

$$\frac{d\,(PV)\,/dr}{PV} \simeq -D$$

Because of its usefulness, the expression

$$-\frac{D}{1+r}$$

is used in its own right, and is sometimes called the *modified duration* measure.

Internal Rate of Return

Internal Rate of Return

We now introduce a concept that sheds further light on the present value rule.

The *internal rate of return* (IRR) of a cash flow stream is the discount rate that makes the present value of the cash flow stream equal to zero. In other words, for a cash flow stream given by

$$C_0, C_1, C_2, \cdots, C_T$$

the internal rate of return is the discount rate r that solves

$$C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_T}{(1+r)^T} = 0$$

The internal rate of return is closely related to the notion of the *yield* on a bond. The yield on a bond is the discount rate that sets the price of the bond equal to the discounted value of the coupon payments and principal.

If C_0 is the negative cashflow entailed by buying the bond ($C_0 = -P$), and C_t is the coupon at date t, then the internal rate of return is the yield on this bond.

$$P = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_T}{(1+r)^T}$$

Mutually exclusive projects. Many investment decisions are concerned with mutually exclusive choices, where if you choose one, you rule out the others. An example is the choice of a replacement machine on a production line. There are many alternatives to choose from, but you can only replace the existing machine with one of the alternatives.

Consider the following rule to guide investment for such cases.

IRR rule (?). Choose the project with the highest IRR.

This rule sounds attractive superficially, since it advocates choosing the project with the "highest return". But, language is slippery. Highest IRR does not imply highest net present value per dollar invested.

Let us consider an example. This example comes from Irving Fisher's book, *The Theory of Interest*, first published in 1930. The numbers have been adapted, but otherwise the example is faithful to Fisher's original story.

Suppose there is a plot of land that is on sale today for 100. There are three possible uses of the land: for **farming, forestry** or **mining**.

Farming generates a steady cash flow of 25, and the cash flow starts immediately. Forestry does not generate any cash for 11 years, while the trees mature. Thereafter, it generates a cash flow of 60. Mining generates high cash flow of 50 for the next five years, but then the land is useless and generates no cash.

	Farming	Forestry	Mining
Year 0	-100	-100	-100
1	25	0	50
2	25	0	50
3	25	0	50
4	25	0	50
5	25	0	50
6	25	0	0
$\vdots$	25	0	0
10	25	0	0
11	25	60	0
12	25	60	0
thereafter	25	60	0

The cash flow from farming is a perpetuity. The net present value of farming at discount rate r is

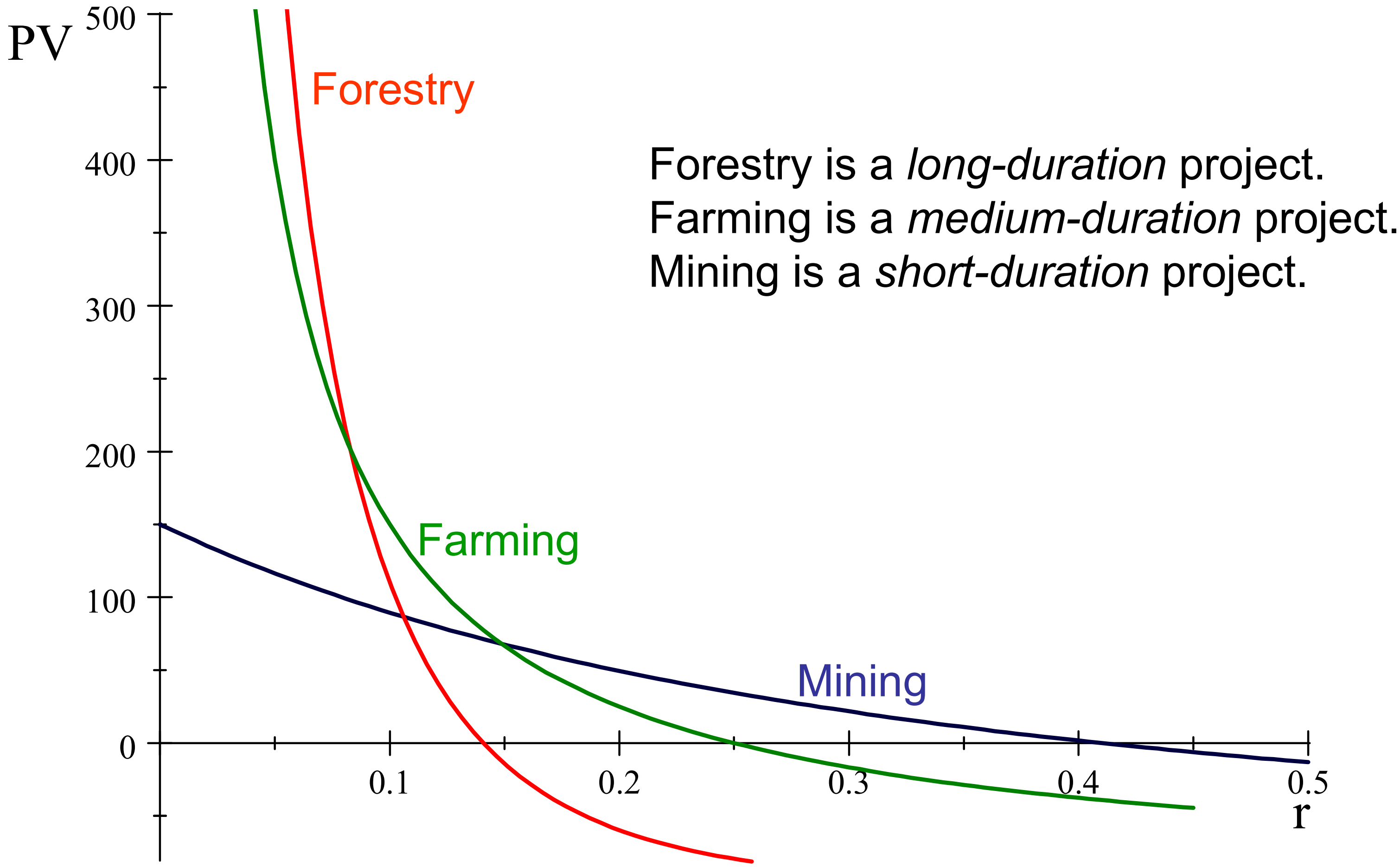
$$-100 + \frac{25}{r}$$

The cash flow from forestry is also a perpetuity, but the cashflows are delayed by 11 years. The net present value of forestry is

$$-100 + \frac{60}{r} \cdot \frac{1}{(1+r)^{11}}$$

The cash flow from mining is an annuity, with a 5 year term and per period cash flow of 50. The net present value of mining is given by

$$-100 + \frac{50}{r}\left(1 - \frac{1}{(1+r)^{5}}\right)$$


PV
500
400
300
200
100
0
0.1
0.2
0.3
0.4
0.5
r
Forestry
Farming
Mining
Forestry is a *long-duration* project.
Farming is a *medium-duration* project.
Mining is a *short-duration* project.

The internal rate of return (IRR) for a project is the discount rate r that makes the net present value equal to zero. For **farming**, the IRR is the discount rate r that solves

$$-100 + \frac{25}{r} = 0$$

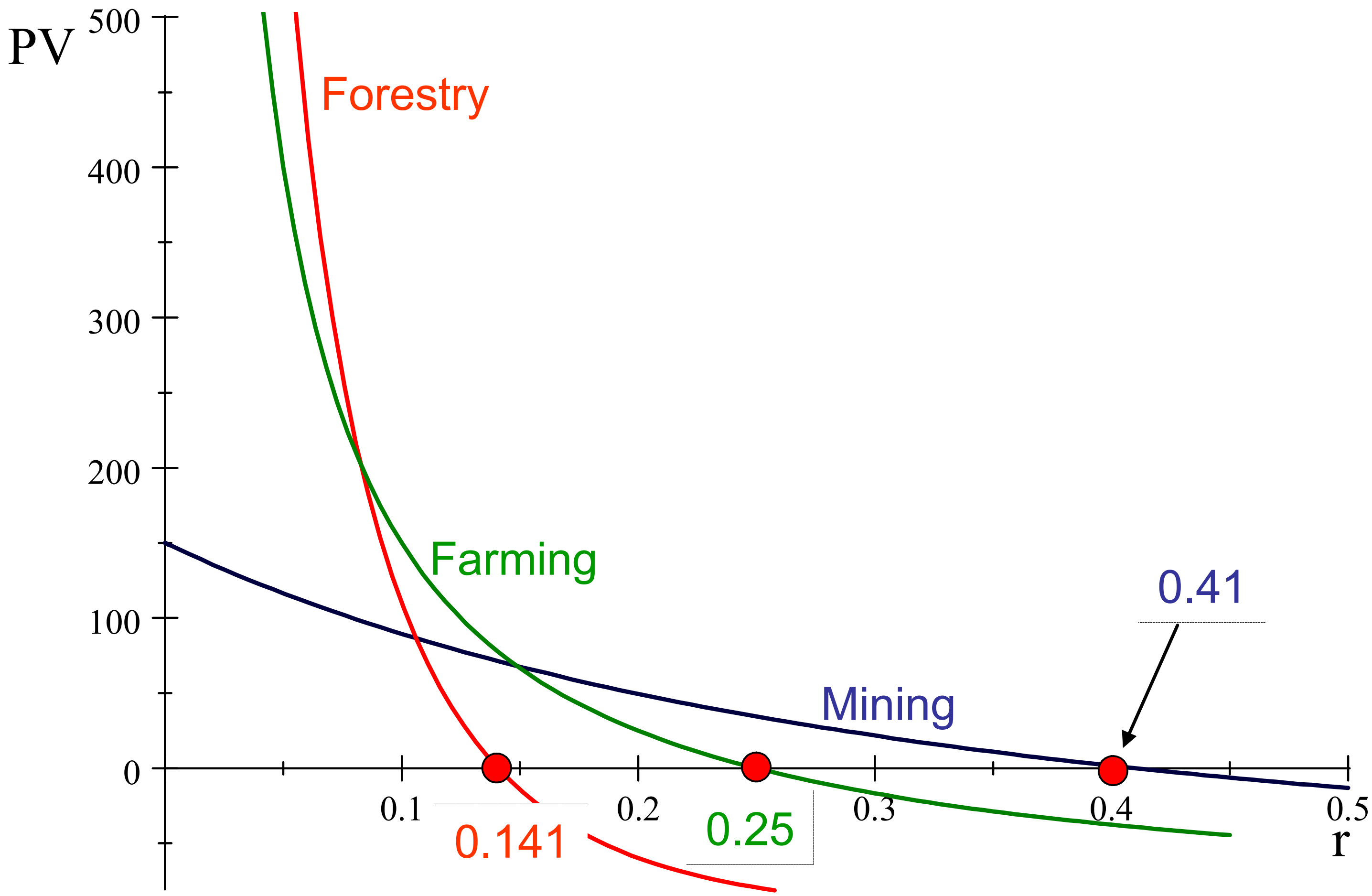
This gives $\text{IRR}_{\text{FARM}} = 25\%$. For **forestry**, the IRR is the discount rate r that solves

$$-100 + \frac{60}{r} \cdot \frac{1}{(1+r)^{11}} = 0$$

This gives $\text{IRR}_{\text{FOREST}} = 14.1\%$. For **mining**, the IRR is the discount rate r that solves

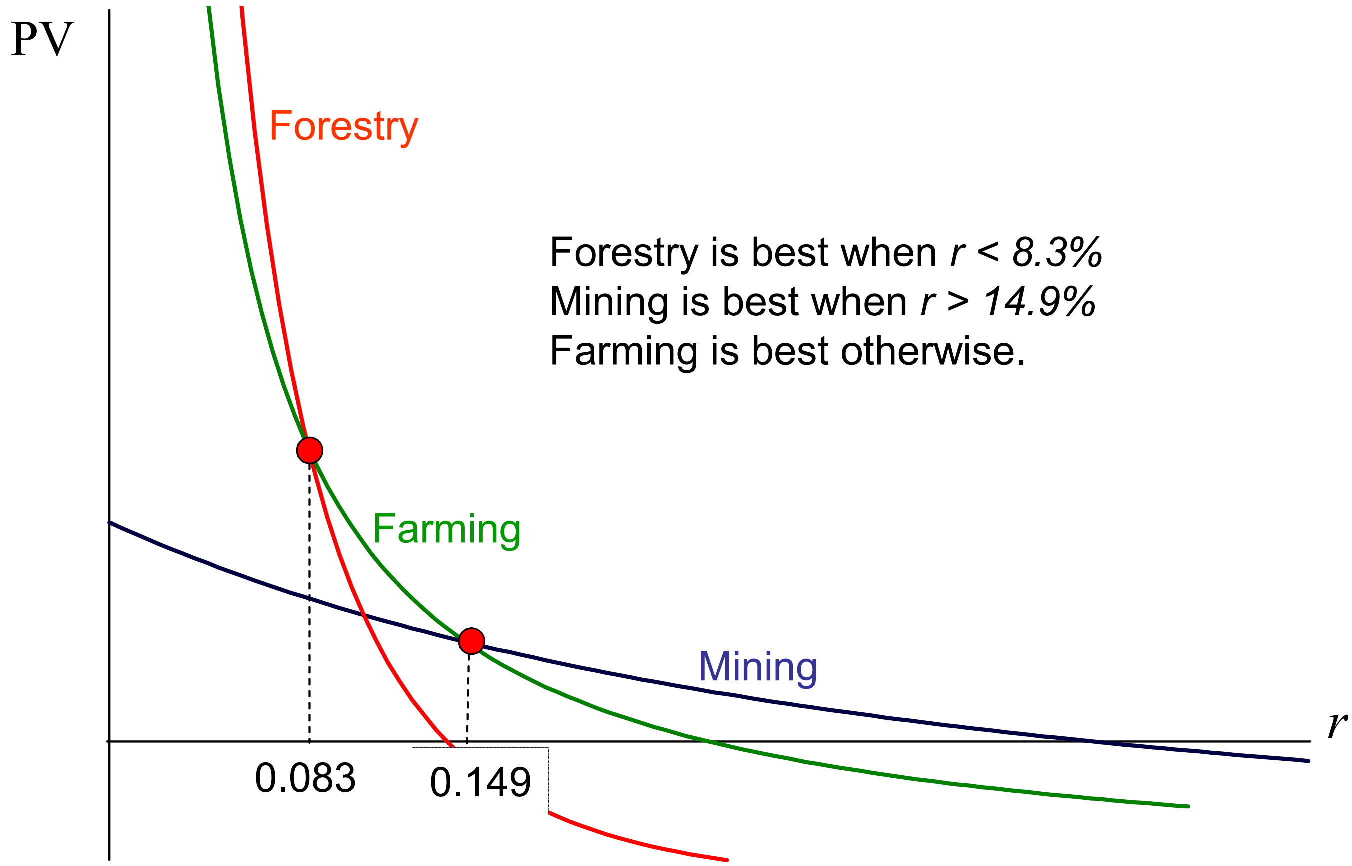
$$-100 + \frac{50}{r}\left(1 - \frac{1}{(1+r)^{5}}\right) = 0$$

This gives $\text{IRR}_{\text{MINE}} = 41\%$. Mining has the highest IRR, followed by farming, with forestry last.



PV
500
400
300
200
100
0
Forestry
Farming
Mining
0.41
0.1
0.2
0.3
0.4
0.5
0.141
0.25
r

However, the ranking of the three uses of land in terms of their net present value depends on r.



This example shows that maximizing the internal rate of return is not the same as maximizing the net present value per dollar invested.

All three projects involve the investment of 100. Their net present values depend on the discount rate r. At low discount rates, long-duration projects such as forestry does best. At high discount rates, short-duration projects like mining does best. At intermediate discount rates, intermediate-duration projects like farming does best. Fisher's example has a modern resonance, too.

◦ Technology stocks are like forestry. They could have high cash flow in the distant future.
◦ Utility stocks are like farming. They have steady cash flow.
◦ Tobacco stocks are like mining. Tobacco stocks have high yield, and so high cash flow. But as tobacco use dwindles, the cash flow will dwindle/cease some time in the near future.

Having learned our lesson about IRR, suppose we modify our rule as follows.