

Proposing New Equilibrium Concepts in Dynamic Games with Noisy Signals*

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In this study, we consider dynamic games with noisy signals in which any signal is possible in equilibrium due to noises in observations. We argue that in games with noisy signals, the consistency condition of beliefs required by the perfect Bayesian equilibrium or sequential equilibrium is too strong to comprehend reasonable outcomes. Accordingly, we propose alternative solution concepts called ε -perfect Bayesian equilibrium and limit perfect Bayesian equilibrium. The two equilibrium concepts rely on a weaker consistency condition that requires beliefs to be updated by Bayes' law only if the likelihood that a signal occurs given that the equilibrium action was played exceeds $\varepsilon > 0$. Our concepts are consistent with empirical observations that show higher deviations from Bayes' law for rare events. We show that under some conditions, both ε -PBE and limit PBE can fully recover the first-mover advantage that disappears with even a slight noise in observing the first mover's strategy.

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I. Introduction

Observability is one of the central issues in noncooperative game theory. In particular, in dynamic games, an equilibrium outcome must be highly sensitive to whether a player can observe the previous action of the other player. In dealing with analytic difficulties that arise from unobservability, the concept of belief (about what a player cannot observe) plays a crucial role. However, almost all equilibrium concepts, including sequential equilibrium (Kreps and Wilson, 1982) and perfect

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Bayesian equilibrium (PBE) (Fudenberg and Tirole, 1991a), impose the minimum requirement of beliefs; this concept is known as weak consistency, which is part of the definition of the perfect Bayesian equilibrium.¹ Weak consistency requires that beliefs must be consistent with equilibrium strategies in a Bayesian sense, i.e., beliefs must be updated according to Bayes' law whenever possible.² This requirement is the weakest version of consistency that has been known thus far. In this study, we argue that this version of consistency is too strong to comprehend all reasonable outcomes.³

Contrary to what we have described, observability is not an all or nothing situation. Perfect observability and perfect unobservability are not the only possibilities in the real world; nevertheless we are confined to the two extreme cases most of the time.⁴ Observations by players are often imperfect. They may be able to observe only a noisy signal of what the other player did. We call such a situation with imperfect observation a dynamic game with noisy signals.⁵ It is a game in which the outcome that the second mover (player II) observes provides only an imperfect signal on the private information of the first mover's (player I) choice, which is relevant to the payoff of the second mover.⁶ In such games, the second mover can infer the private information of the first mover stochastically by observing the signal that occurs as a result of his action if he is informed of the probabilistic signal-generating rule. However, if the second player knows what the equilibrium strategies are, in other words, what strategy the first player is supposed to play in equilibrium, the inference should be modified by this additional information. For example, player II happens to observe an outcome that can occur only with very low

¹ Mas-Colell *et al.* (1995) distinguished PBE and weak PBE, and Fudenberg and Tirole (1991a) defined PBE by imposing the condition of not signaling what one does not know on the off-the-equilibrium belief in addition to the requirement of weak PBE. However, we do not distinguish between the two concepts. The additional requirement is not binding in games we consider in this study.

² What "whenever possible" means exactly has never been precisely defined. Watson (2017) put forward a recent study in this direction. He requires conditional-probability updating on separate dimensions of the strategy space.

³ By a reasonable outcome, we mean a self-enforcing agreement based on a reasonable belief, which will be redefined in this paper.

⁴ Fudenberg and Tirole (1991a) defined the perfect Bayesian equilibrium only for games with perfectly observed actions. However, their definition can be straightforwardly extended to games with imperfectly observed actions.

⁵ Our model of noisy signals should be distinguished from noisy signaling models by Matthews and Mirman (1983) and Hertzendorf (1993) in the sense that no noisy signaling is involved in our model. The noisy signals in our model are just a natural consequence and not a consequence of a player's strategic choice to influence the other player's belief.

⁶ In this sense, dynamic games with noisy signals share common features with games of imperfect public information. See Fudenberg and Tirole (1991b) for examples of games of imperfect public information. One difference is that signals in our games need not be public information. They may be private information of the second mover.

probability if player I takes the equilibrium action. The notion of weak consistency requires player II to still believe in the probability that player I took the equilibrium action, insofar as the outcome he observed can be generated from the equilibrium action.⁷ However, what if this outcome can occur with much higher (*ex ante*) probability⁸ as a result of deviating from the equilibrium? Is it more reasonable to still believe that player I took the equilibrium action but the unlikely outcome has occurred unexpectedly, rather than the first mover took a non-equilibrium action which was very likely to lead to the observation?

In a game with perfect observability, an equilibrium action induces an outcome that is possible in equilibrium, whereas an off-the-equilibrium action leads to an outcome that is impossible in equilibrium. Therefore, player II naturally believes that player I took an equilibrium action if he observes an outcome that is possible only on an equilibrium path. However, in a game with imperfect observability, which can occur mainly due to noise (with unbounded support), any outcome is possible, whether player I took an equilibrium action or otherwise. Therefore, player I did not necessarily take an equilibrium action when player II observed an outcome that was possible on an equilibrium path. It could also be possible if player I took an off-the-equilibrium action. Moreover, contrary to games with perfect unobservability in which some outcome is possible in equilibrium and some other outcomes are not, every information set (i.e., every outcome) can be reached on an equilibrium path in games with imperfect observability (due to noisy signals). However, the probabilities of reaching different information sets may differ, depending on whether an equilibrium action is selected. Information about this probability difference is completely ignored by PBE, which may have an important implication.⁹ Then, when should we believe that player II reached a certain information set as a result of an off-the-equilibrium action by player I? If the probability that the information set is reached given that player I took an equilibrium action is very low, rejecting the hypothesis that player I selected the proposed equilibrium action and believing that player I deviated from the equilibrium strategy may be more reasonable. At least, we should doubt the assumption that she took the equilibrium strategy. In this event, we regard the outcome as an almost off-the-equilibrium signal.

⁷ This case occurs only when we take only pure strategies into account.

⁸ By *ex ante* probability, we mean the probability at the point of time before the players obtain additional information of what the equilibrium strategies are, i.e., what strategies they will agree to play in equilibrium.

⁹ For example, h_1 and h_2 are two information sets. We suppose the probability that h_1 is reached if player I took an equilibrium action is low and the probability is high if she did not take an equilibrium action. On the other hand, we suppose the probability that h_2 is reached is high if player I took an equilibrium action and it is low if she did not take an equilibrium action. Then, the beliefs that player I took an off-the-equilibrium action at h_1 and h_2 should be updated differently, more specifically, the probability at h_1 should be higher.

For instance, a worker (player I) exerts effort (a), and an outcome (y) is realized as a result of it. In this scenario, we assume that y can take the value of a with a probability of .9999 and of $a-1$ with a probability of .0001 if player I selects action a . Then, a^* is the equilibrium effort. If a principal (player II) observes $y = a^* - 1$, can player II reasonably believe that player I chose a^* ? The perfect Bayesian equilibrium stipulates that the unique reasonable belief is to assume the probability that player I took a^* , not $a^* - 1$. However, believing that player I chose action $a^* - 1$ seems more reasonable (at least) to us because, given that $a = a^*$, the likelihood (probability) that $y = a^* - 1$ actually happens is very small (.0001).

On this basis, we think that insisting on the weak consistency of beliefs that the perfect Bayesian equilibrium requires is too demanding. The perfect Bayesian equilibrium imposes an overly strong restriction on beliefs on the equilibrium path, which occurs only with infinitesimally small probability (too stringent), whereas it imposes no restrictions on the path that occurs in equilibrium with zero probability, thereby allowing arbitrary beliefs (too lenient). Furthermore, the PBE requires that one assign the same beliefs on the path that can occur only with a very low probability in equilibrium as on the path that can occur with a very high probability in equilibrium. It seems reasonable, neither. For example, player I intended to mean “reminder” through the word “memento” used in equilibrium. However, if the second mover observes a noisy signal “mementa” or “memento,” he may believe that the first mover who intended to mean “reminder” only misspelled it because such misspelling often occurs. However, if he observes a signal “momentu,” the second mover will naturally believe that the first mover intended the word “momentum” because those who intended to spell “memento” are very unlikely to misspell it as “momentu.” In this case, it might be more reasonable to believe that she intended to mean a different word “momentum.”¹⁰

On the basis of this consideration, we propose two equilibrium concepts, the ε -perfect Bayesian equilibrium (ε -PBE) and the limit perfect Bayesian equilibrium (limit PBE),¹¹ both of which are slightly weaker than the perfect Bayesian

¹⁰ Note that any possible word including “memento” and “momentum” can be an equilibrium message in this case. A reader also commented that when “momentu” is observed, it is unlikely that she intended “bomentu” because “bomentu” is not a word. We agree that beliefs should be restricted to the set of possible states. In this case, “bomentu” is not a possible state.

¹¹ Similarly, Myerson and Reny (2019) defined the perfect conditional ε -equilibrium to extend sequential equilibrium to games with infinite sets of signals and actions. They first defined the conditional ε -equilibrium by a strategy profile such that no player could expect significant gains by unilaterally deviating from it after any event that has positive probability in the equilibrium. Then, they defined the perfect conditional ε -equilibrium using a strategy profile such that a pair of a perturbed strategy profile and nature’s perturbation in the neighborhood exist, which constitutes a conditional ε -equilibrium. Our definition entails applying consistency to an event with a positive *likelihood*, not to an event with a positive *probability*. This notion is reasonable because a positive

equilibrium. The notion of ε -PBE imposes the consistency condition only if the likelihood that a noisy signal occurs given that the equilibrium action was played is reasonably high (higher than ε for some small $\varepsilon > 0$) and allows the second mover to believe that the first mover deviated from the proposed equilibrium if the likelihood is not higher than $\varepsilon > 0$.¹² We call this consistency ε -likelihood consistency. The rationale for ε -likelihood consistency is that if the likelihood is very low, the second player will reasonably believe that the first mover made a mistake rather than believing that the second mover himself made a wrong observation, thereby ignoring the equilibrium message.¹³ If an equilibrium is an ε -PBE for any small $\varepsilon > 0$,¹⁴ we call it a limit PBE.

These concepts turn out to be very useful, especially in games with noisy signals that make every outcome possible in equilibrium. By allowing additional natural beliefs on the equilibrium path rather than restricting beliefs off the equilibrium path, we can obtain realistic outcomes in equilibrium in many games with noisy signals. For example, Bagwell (1995) argued that the first-mover advantage is eliminated if even a slight amount of noise exists in observing the first-mover's choice, insofar as attention is restricted to pure strategies; however, this scenario can be approached by a mixed strategy equilibrium.¹⁵ However, we show that ε -PBE enables the first-mover advantage to be fully recovered. Our result of recovering the first-mover advantage fully complements and reinforces the research of Bagwell (1991) and van Damme and Hurkens (1997), which states that the first-mover advantage can be approached by a mixed strategy equilibrium. Our result is a

likelihood enables us to update the posterior belief by the conditional density function (not the conditional probability).

¹² Ortoleva (2012) proposed a different equilibrium concept of hypothesis testing equilibrium from a similar motivation. However, his approach non-Bayesian. He assumed that players use subjective priors, which were initially selected by them among all possible priors, and pick a new prior if they receive an unexpected evidence. For more details of the hypothesis testing equilibrium, see Ortoleva (2012) or Sun (2019).

¹³ Some may think that if this ε -likelihood event occurs, believing that the first mover chose an action maximizing the likelihood function rather than just ignoring the equilibrium message may be more reasonable. If so, the resulting equilibrium concept may not contain the set of PBE. Ortoleva (2012) took the maximum likelihood approach.

¹⁴ A likelihood and a (posterior) belief are both conditional probabilities. However, a belief is a probability of reaching a node at an information set conditional on the event that the information set is reached, whereas a likelihood is a probability of reaching an information set conditional on the event that the player chose an equilibrium action. Additionally, a likelihood is similar to the concept of plausibility used by Bonanno (2013). However, plausibility is defined for all histories and not for information sets.

¹⁵ Maggi (1999) showed that the first-mover advantage reappears when the first mover's choice is based on pure private information, i.e., private information that is payoff-irrelevant for the second mover. By focusing only on "pure private information", he abstracted from signaling consideration. If one considers private information that is not pure, the first-mover advantage may or may not appear. See Gal-Or (1987), Matthews and Mirman (1983) and Hertzendorf (1993).

positive one, i.e., recovering the first-mover advantage (fully) is possible even in pure strategies, whereas their results based on the concept of PBE are negative, especially when the noise is not negligible.

A large body of empirical literature has argued that players tend to violate Bayes' law.¹⁶ For instance, Grether (1992) showed that empirical data for rare events are more likely to deviate from Bayes' law. Holt and Smith (2009) also presented experimental evidence that players do not tend to follow Bayes' law after very unlikely events occur.

However, the almost-off-the-equilibrium paths on which we focus in this study are not simply rare events that are unlikely but events that have low conditional likelihood given a certain hypothesis (that player I actually selected some action that is not observable). If some event that was unexpected from what a player believed (i.e., somewhat contradictory to his initial belief) occurs, his response can be either to ignore this rare event (with maintaining his initial belief) or to take the event seriously but doubt the initial belief. If he is a Bayesian decision maker, he will take the first approach by regarding the new information as exceptional and simply ignoring it. If he is almost Bayesian but not perfectly Bayesian, he will take the second approach by doubting the initial belief and placing more weight on the new evidence. For example, we imagine a situation in which a vaccine for a pandemic is developed. We assume that all the clinical trials for the vaccine are completed successfully and the vaccine is believed to be highly safe. However, if a vaccinated person dies of serious blood clots, some may regard it as an exceptional event that can be attributed to his underlying disease. Meanwhile, others may doubt his initial belief that the vaccine is safe. The responses vary across ages, genders, education levels, regions, etc.¹⁷

The organization of this paper is as follows. In Section 2, we introduce the definitions of ε -PBE and limit PBE. Then, in Section 3, we revisit the example by Bagwell (1995) and demonstrate that we can recover the first-mover advantage by applying ε -PBE and generalize this claim. In Section 4, we apply the two concepts to a price competition model with a Stackelberg leader. Concluding remarks follow in Section 5.

¹⁶ See, for example, Kahneman and Tversky (1973), Grether (1992), Griffin and Tversky (1992), and Holt and Smith (2009).

¹⁷ For example, in the United Kingdom, 27% were reported to be hesitant. Meanwhile, in Canada, 14% reported unlikely to obtain the COVID-19 vaccine when available. Additionally, in China, the rate of vaccination among individuals with a master's degree or above was 37.8% compared with 20.3% among those with a high school or below education level. See Wang *et al.* (2021).

II. Definitions

The scenario in our study has two players, namely, player I (“she”) and player II (“he”). Player I first selects an action a_1 from a set $A_1 \subset \mathbb{R}$. Each action induces a probability distribution over observable outcomes y in a set $Y \subset \mathbb{R}$. $f(y|a_1)$ and $F(y|a_1)$ denote the density function and the corresponding probability distribution function conditional on the action a_1 . After observing y (not observing a_1), player II responds by selecting an action a_2 from a set $A_2 \subset \mathbb{R}$. We assume that $f(y|a_1)$ is common knowledge. The support of f given a_1 is defined by $\text{supp}(f|a_1) = \{y | f(y|a_1) > 0\}$. Most of the time, we will assume that $A_1 = Y = \mathbb{R}$, i.e., $\text{supp}(f|a_1) = \mathbb{R}$ for any $a_1 \in \mathbb{R}$. We also assume that $f(y|a_1)$ first-order stochastically dominates $f(y|a'_1)$ if $a_1 > a'_1$.

The payoff to player i is given by a function $U^i : A_1 \times A_2 \rightarrow \mathbb{R}$. We assume that U^i is twice-continuously differentiable with respect to a_1 and a_2 and that it is concave in a_1 and a_2 . This configuration guarantees that for any a_j , U^i is uniquely maximized by $a_i^{BR}(a_j)$, which gives continuous best response functions. We also assume that $a_i^{BR}(a_j)$ is strictly monotonic, thus implying that we exclude the possibility that $U_{ij}^i = 0$.¹⁸ Notably, U^i does not depend directly on y .¹⁹

A strategy for player I, σ^I , is identical to her action. A system of beliefs is defined by a map from the set of possible observations (Y) to $\Delta(A_1)$, where ΔA_1 denotes the set of all probability distribution functions (or density functions) over A_1 . We use the notation $b(y) = a_1^0$ for the belief, where $b : Y \rightarrow A_1$ is a belief function. We assume that the belief function is continuous. Finally, a strategy for player II is a function from the set of beliefs (A_1) into A_2 , $\sigma^{II} : A_1 \rightarrow A_2$. Notably, σ^{II} depends on y only indirectly by forming belief b because we assume that y does not affect U^{II} directly.

An assessment is a pair (σ, b) of a strategy profile σ and a system of beliefs b , where $\sigma \equiv (\sigma^I, \sigma^{II}) = (a_1, a_2(b))$. σ^I and σ^{II} are both defined as pure strategies. For simplicity, we will restrict our attention to pure strategies throughout the study because our primary purpose of this research is to recover the Stackelberg outcome

¹⁸ If $U_{ij}^i = 0$ for all a_1 and a_2 , we will say that a_1 and a_2 are strategically independent. In this case, $a_i^{BR}(a_j)$ is the same for any a_j , that is, the best response function $a_i^{BR}(a_j)$ is constant. We ignore this possibility because no strategic interaction occurs in this case. If $U_{ij}^i \geq 0$ for any a_1 and a_2 , a_1 and a_2 are called strategic complements (strategic substitutes, *resp.*). Our analysis encompasses both cases.

¹⁹ We can imagine many situations in which the payoffs of the players depend directly on y . For instance, in a Cournot game with fluctuating demands, the profit of each firm depends directly on the market price. The output produced by the rival firm affects the profit only indirectly by affecting the market price. Additionally, in a principal–agent game, the payoff of the principal usually depends directly on the output. The agent’s effort level affects the principal’s utility only through determining the output level stochastically. Our equilibrium concepts can be applied straightforwardly to those situations. In other words, this assumption is not crucial but just for expositional convenience.

(first-mover advantage).

Before we propose the two definitions of our main equilibrium concepts, we introduce the definition of the perfect Bayesian equilibrium, which is adapted to our model.

Definition 1. An assessment $(a_1^*, a_2^*(b), b(y))$ is a perfect Bayesian equilibrium (PBE) if (i) it is sequentially rational, i.e., $\int_Y U^I(a_1^*, a_2^*(b(y)))f(y|a_1^*)dy \geq \int_Y U^I(a_1, a_2(b(y)))f(y|a_1)dy$, $\forall a_1 \in A_1$ and for every $y \in Y$, $U^II(b(y), a_2^*(b(y))) \geq U^II(b(y), a_2)$, $\forall a_2 \in A_2$ and (ii) $b(y)$ is weakly consistent, i.e., $b(y) = a_1^*$ if $y \in \text{supp}(f|a_1^*)$ and $b(y)$ is an arbitrary density function if $y \notin \text{supp}(f|a_1^*)$.

To formalize our solution concepts, we need to define a slightly relaxed notion of consistency as follows.

Definition 2. An assessment $(a_1^*, a_2^*(b), b(y))$ satisfies ε -likelihood consistency if for any y , $b(y) = a_1^*$ if $L(a_1^*; y) > \varepsilon$ and $b(y) \in \mathbb{R}$ can be arbitrary if $L(a_1^*; y) \leq \varepsilon$, where $L(a_1; y) = f(y|a_1)$ is a likelihood function.

The likelihood function tells us how likely the occurrence of y is if player I chooses the equilibrium action a_1^* . We may call this scenario $(L(a_1^*; y) \geq \varepsilon)$ a simple likelihood test. If $L(a_1^*; y) \leq \varepsilon$, we reject the hypothesis that player I is playing an equilibrium strategy.²⁰ In this definition, we slightly extend the meaning of “off the equilibrium” to the case that a random outcome y such that $L(a_1^*; y) \leq \varepsilon$ occurs. This event in which $L(a_1^*; y) \leq \varepsilon$ can be interpreted as almost off the equilibrium path.²¹ $L(a_1^*; y) \leq \varepsilon$ if $y \notin \text{supp}(f|a_1^*)$. Therefore, weak consistency implies ε -likelihood consistency. We interpret an equilibrium event and an off-the-equilibrium event in terms of likelihood, not in terms of probability. Even if $L(a_1^*; y) = f(y|a_1^*) > 0$, the probability that a particular value of y occurs given that a_1^* is chosen is zero if y is a continuous random variable without any atom that supports its probability distribution. We believe that this likelihood approach is more relevant to defining an equilibrium event. If the likelihood $L(a_1^*; y)$ is positive (although $\mathbb{P}(y|a_1^*) = 0$), the conditional density function (belief) can be updated by using the likelihood function, which is the most crucial

²⁰ Alternatively, some may want to use a likelihood ratio test, but it is not appropriate in this situation to determine whether player I played the equilibrium action or deviated to some action among many possible off-the-equilibrium actions. Additionally, the test is problematic because the likelihood ratio does not reflect the information that the equilibrium action is more often used than nonequilibrium actions. Therefore, comparing the likelihood without weights will not make much sense. Nevertheless, this concept is a promising research direction, as it allows almost off-the-equilibrium beliefs $b(y)$ only if $L(a_1^*; y) \leq \varepsilon$ and $L(b(y); y)/L(a_1^*; y) \geq K$ for some large K .

²¹ We can define an almost off-the-equilibrium event in terms of either the likelihood is less than ε or the p value is less than ε interchangeably.

aspect for an equilibrium event.²² Now, we can define our solution concepts formally.

Definition 3. An assessment $(a_1^*, a_2^*(b), b(y))$ is an ε -perfect Bayesian equilibrium (ε -PBE) for some $\varepsilon > 0$ if it satisfies sequential rationality and ε -likelihood consistency.

This concept, ε -PBE, is similar to ε -perfect equilibrium, which can be roughly defined by a strategy profile satisfying the property that if a certain pure strategy yields a strictly lower payoff than another, the strategy should be used with a probability less than $\varepsilon (> 0)$, not necessarily with a zero probability.²³ ε -PBE is a slight departure from the weak consistency of a *belief*, whereas ε -perfect equilibrium is a slight departure from the best response of a *strategy*. Just as the trembling hand perfect equilibrium is defined as the limit of ε -perfect equilibrium, we can define a stronger equilibrium concept that can be obtained by making ε approach zero.

Definition 4. A strategy profile, $(a_1^*, a_2^*(b))$, is a limit perfect Bayesian equilibrium (limit PBE) if for any $\varepsilon > 0$ such that $\varepsilon \leq \bar{\varepsilon}$ for some $\bar{\varepsilon} > 0$, $b(y; \varepsilon)$ exists such that an assessment $(a_1^*, a_2^*(b), b(y; \varepsilon))$ is an ε -PBE.

The difference between the definitions of the ε -PBE and the limit PBE is that the former holds for some small $\varepsilon > 0$ while the latter holds for any small $\varepsilon > 0$. Although a perfect equilibrium is a limit of ε -perfect equilibrium,²⁴ a limit PBE is not necessarily a limit of ε -PBE²⁵ because the definition of the limit PBE does not require that $\lim_{\varepsilon \rightarrow 0} b(y; \varepsilon)$ exists.²⁶ If we require that $b(y)$ must exist such that $b(y) = \lim_{\varepsilon \rightarrow 0} b(y; \varepsilon)$,²⁷ this additional continuity requirement is so strong that the

²² Myerson and Reny (2019) defined perfect conditional ε -equilibrium, which is conditional on positive probability events, not conditional on positive likelihood events.

²³ See Myerson (1978) for the formal definition of the ε -perfect equilibrium.

²⁴ The trembling hand perfect equilibrium requires a_2^* to be the best response to any nearby others' strategies perturbed from the equilibrium strategies (which can also be interpreted as a sequence of beliefs). However, the sequential equilibrium only requires a_2^* to be a best response to the limit of the belief sequence. Therefore, limit PBE is in its spirit closer to the trembling hand perfect equilibrium than the sequential equilibrium in the sense that it requires $a_2(b)$ to be a best response to $b(y; \varepsilon)$, not to $b(y)$, the limit of $b(y; \varepsilon)$.

²⁵ In evolutionary game theory, the concept of the limit ESS is defined from a similar motivation. It was proposed by Selten (1983) to alleviate the severe nonexistence problem of ESS. Contrary to the limit PBE, however, the limit ESS is required to be a limit of a sequence of strict ESS in ε -perturbed games. However, it need not be ESS in the limit.

²⁶ As long as we restrict our attention to pure strategies, $\Delta(A_i) = A_i \subset \mathbb{R}$. Therefore, we can use the uniform topology induced by the uniform metric on \mathbb{R}^Y .

²⁷ Given that we assume that the belief function $b(y)$ is continuous, we require uniform

resulting equilibrium concept, which we call strong limit PBE, becomes equivalent to PBE, just as a perfect equilibrium is a limit of ε -perfect equilibrium.

Definition 5. An assessment $(a_1^*, a_2^*(b), b(y))$ is a strong limit perfect Bayesian equilibrium (strong limit PBE) if for any $\varepsilon > 0$ such that $\varepsilon \leq \bar{\varepsilon}$ for some $\bar{\varepsilon} > 0$, $b(y; \varepsilon)$ exists such that (i) $\lim_{\varepsilon \rightarrow 0} b(y; \varepsilon) = b(y)$ uniformly and (ii) $(a_1^*, a_2^*(b), b(y; \varepsilon))$ satisfies ε -PBE.

We have the following properties, i.e., inclusion relations among those equilibrium concepts. The proofs are provided in the Appendix.

Proposition 1. $PBE \subset \text{limit PBE} \subset \varepsilon\text{-PBE}$.

Proposition 2. The strong limit PBE is equivalent to PBE.

As we discussed above, the difference among the three equilibrium concepts in Proposition 1 lies in the required notion of consistency. Given that ε -PBE, limit PBE, and PBE allow arbitrary beliefs only if $L(a_1^*; y) \leq \varepsilon$ for some $\varepsilon > 0$, $L(a_1^*; y) \leq \varepsilon$ for any small $\varepsilon > 0$, and $L(a_1^*; y) = 0$, respectively, PBE is the strongest, and ε -PBE is the weakest. The intuitive reason for the equivalence between the strong limit PBE and PBE is also clear. If $\lim_{\varepsilon \rightarrow 0} b(y; \varepsilon) \neq b(y)$, as in a limit PBE, a different equilibrium from PBE may exist, which requires the belief $b(y)$. However, if $\lim_{\varepsilon \rightarrow 0} b(y; \varepsilon) = b(y)$, as in the strong limit PBE, no such occurrence is possible, thus implying that the resulting equilibrium must coincide with the PBE.

In the subsequent sections, we will present some counterexamples for the cases that the converses do not hold, i.e., ε -PBE does not imply limit PBE and that limit PBE does not imply PBE.

III. Stackelberg Model

We consider an example by Bagwell (1995) that captures the main feature of the Stackelberg model (Figure 1). In this example, the unique Nash equilibrium is (C, C) in a static game (in which neither player can observe the other's choice), while the unique subgame perfect outcome is (S, S) in a sequential game (in which player II can observe player I's choice), thus yielding the first-mover advantage.

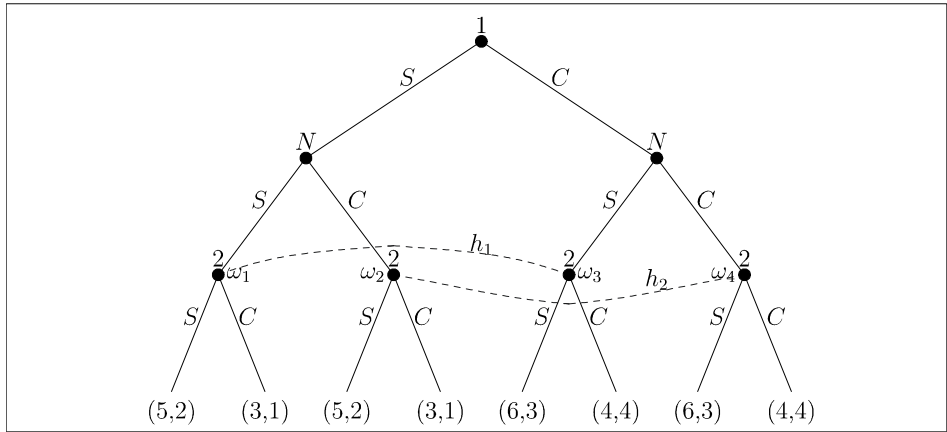
convergence of $b(y; \varepsilon)$ to $b(y)$ because the pointwise limit of continuous functions may not be continuous.

We suppose that player II can observe player I only imperfectly, i.e., if player I selects S , player II may receive a signal of either S with probability $1-\delta$ or C with probability $\delta(>0)$. If player I chooses C , player II receives a noisy signal of either C with probability $1-\delta$ or S with probability δ . The extensive form of the resulting game is illustrated in Figure 2. Given the imperfect observation, any information set can be reached after player I's choice of either S or C .

[Figure 1] Bagwell's Example

		II	
I		S	C
		(5, 2)	(3, 1)
	S	(5, 2)	(3, 1)
	C	(6, 3)	(4, 4)

[Figure 2] Extensive Form of Bagwell's Example



3.1. Pure Strategy Equilibrium

The unique pure strategy PBE outcome of this imperfect information game is (C,C) . To see that (C,C) is an equilibrium outcome, the weak consistency condition pins down the posterior belief at two information sets, h_1 and h_2 , as $(\mu(\omega_1 | h_1), \mu(\omega_3 | h_1)) = (0,1)$ and $(\mu(\omega_2 | h_2), \mu(\omega_4 | h_2)) = (0,1)$. $\mu(\omega | h_i)$ is the posterior probability that player II reaches node ω given that he reaches information set h_i because he believes that she took the action C regardless of his observation. Thus, the optimal move for player II is to choose C at both information sets h_1 and h_2 . Therefore, player I selects C .

The Stackelberg outcome (S,S) is not a PBE. If it is an equilibrium outcome, the only consistent belief is that $((\mu(\omega_1 | h_1), \mu(\omega_3 | h_1)), (\mu(\omega_2 | h_2), \mu(\omega_4 | h_2))) = ((1,0), (1,0))$. Then, player II will choose his best response to S , which is S , whether he observes either S or C . In particular, even when he observes C , he

selects S by reasoning that he received a wrong signal because player I must have opted for S . This scenario implies that player I prefers deviating from S to C . Given that this configuration holds for any $\delta > 0$, the first-mover advantage disappears when even a slight noise ($\delta > 0$) exists in the observation of player II.

3.2. Mixed Strategy Equilibrium

Although Bagwell's result provided in Section 3.1 appears to be striking, some authors criticized the result on the grounds that it is a consequence of neglecting to consider mixed strategies, as we mentioned in Section 1. Bagwell showed that the game described in Figure 2 always has a mixed strategy equilibrium that converges to the Stackelberg outcome as the noise vanishes. Furthermore, van Damme and Hurkens (1997) argued that only the mixed strategy equilibrium that converges to the Stackelberg outcome is selected by the tracing procedure proposed by Harsanyi and Selten (1988) and Harsanyi (1995).

To elucidate their arguments, $\sigma \equiv (p, q(y))$ is a mixed strategy profile where p is the probability that player I selects S and $q(y)$ is the probability that player II chooses S given the observation y . Then, the weak consistency condition of PBE requires the posterior belief:

$$\begin{aligned}\mu(\omega_1) &= \frac{p(1-\delta)}{p(1-\delta) + (1-p)\delta} \equiv p_1, \\ \mu(\omega_3) &= \frac{(1-p)\delta}{p(1-\delta) + (1-p)\delta} \equiv p_2, \\ \mu(\omega_2) &= \frac{p\delta}{p\delta + (1-p)(1-\delta)} \equiv p_3, \\ \mu(\omega_4) &= \frac{(1-p)(1-\delta)}{p\delta + (1-p)(1-\delta)} \equiv p_4.\end{aligned}$$

$\sigma^* \equiv (p^*, q^*(y))$ is a mixed strategy equilibrium. Then, $p^* \in (0, 1)$ can be found from

$$2p_3 + 3p_4 = p_3 + 4p_4.$$

Using $p_4 = 1 - p_3$, we obtain $p_3 = \frac{1}{2}$, which leads to $p^* = 1 - \delta$. Additionally, player I's indifference leads to $q^*(C) = \frac{1-3\delta}{2(1-2\delta)}$. Given that $p_1 > p_3$ due to the assumption of first-order stochastic dominance ($1 - \delta > \delta$), player II strictly prefers S to C given the observation $y = S$. Therefore, we obtain a mixed strategy equilibrium $\sigma^*(\delta) = (p^*, q^*(S), q^*(C)) = (1 - \delta, 1, \frac{1-3\delta}{2(1-2\delta)})$. Evidently, $\lim_{\delta \rightarrow 0} \sigma^*(\delta) = (1, 1, \frac{1}{2})$. Allowing mixed strategies does not recover the first-mover advantage fully.

Nevertheless, it almost approaches the Stackelberg outcome. What if we apply our equilibrium concepts to this mixed strategy equilibrium? It satisfies ε -PBE and limit PBE because PBE always implies ε -PBE and limit PBE.

Harsanyi and Selten (1988) proposed the following tracing procedure of a particular equilibrium. For a game G , a linear tracing procedure is defined by a family of games G^t with $t \in [0,1]$ such that

$$U_i^t(\sigma_i, \sigma_{-i}) = tU_i(\sigma_i, \sigma_{-i}) + (1-t)U_i(\sigma_i, \sigma_{-i}^0), \quad (1)$$

where σ_{-i}^0 is an initial belief about σ_{-i} . Then, each player maximizes U_i^t instead of U_i at every t . When $t = 0$, (1) is reduced to $U_i^0(\sigma_i, \sigma_{-i}) = U_i(\sigma_i, \sigma_{-i}^0)$. Therefore, each player simply chooses the best response to his/her initial belief about other players' strategies σ_{-i}^0 . This response is called naive Bayesian behavior. As t increases, each player becomes a more sophisticated Bayesian decision maker. He/she selects the best response by putting additional weight on the original payoff function. Eventually, when $t = 1$, (1) boils down to $U_i^1(\sigma_i, \sigma_{-i}) = U_i(\sigma_i, \sigma_{-i})$, which is the payoff function in the original game G . This procedure can be interpreted as a learning process through which players learn to play a particular equilibrium. The question that remains is where the initial belief (prior) originates. Harsanyi (1995) recommended that the prior should assign higher probability to pure strategies that would be the best replies to the other players' expected strategies in a wide range of strategic situations. By combining the equilibrium selection criteria of Harsanyi and Selten (1988) and Harsanyi (1995), van Damme and Hurkens (1997) showed that only the mixed strategy equilibrium that was derived above is selected, thus implying that the first-mover advantage is almost recovered by the mixed strategy equilibrium. We will not apply this selection criterion to our games because we assume that players know the equilibrium strategies, following the spirit of traditional game theory, which is contrary to the learning approach.

3.3. Main Idea and Results

Intuitively, noise, no matter how small it is, makes any signal possible when player I makes the equilibrium choice S . Hence, no off-the-equilibrium signal emerges. Therefore, regardless of what signal player II may receive, it should be interpreted as the equilibrium meaning. If player I's choice cannot affect player II's belief and his choice due to imperfect observation, the resulting equilibrium outcome must coincide with the static Nash outcome. Given that the outcome yielding the first-mover advantage is not a static Nash equilibrium, it cannot be a PBE. The weak consistency condition that PBE requires ignores the information about the difference in probabilities even if the ex ante likelihood that player I selected S at information set h_1 (when the signal S was observed) is much

higher than the ex ante likelihood at information set h_2 (when the signal C was observed). Thus, the player is forced to assign identical beliefs at h_1 and h_2 by superseding the information that she can infer from equilibrium behavior (which is supposed to be played). The information about a difference in ex ante probabilities should not be ignored but taken seriously. Thus, the posterior beliefs at h_1 and h_2 need not be identical. In particular, if the ex ante likelihood that S was played given that C was observed at h_2 is very small, player II should doubt the presumption that the signal C resulted from player I's equilibrium behavior of choosing S and be open to all other possibilities by considering all possible scenarios. This is the motivation of ε -PBE.

If we use ε -PBE as our solution concept, we can see that the first-mover advantage reappears in equilibrium. We suppose that equilibrium is reached when player I chooses S and player II chooses S at h_1 and C at h_2 . After observing the signal S , player II will believe that player I chose S because S is the equilibrium meaning.²⁸ Therefore, he will respond by choosing S on the basis of the belief $(\mu(\omega_1 | h_1), \mu(\omega_3 | h_1)) = (1, 0)$. However, if he observes the signal C , he will think that player I may select S because it is an equilibrium action. However, he will perceive that the probability of observing C if player I actually chose S is very low, i.e., $f(C|S) = \delta$. If $\delta < \varepsilon$, ε -PBE does not require the belief to be updated by Bayes' law. In this case, the arbitrariness of the belief allows player II to believe that player I chose C , thereby responding by selecting C . Given this knowledge, player I will opt for S because $5(1-\delta) + 4\delta > 5\delta + 4(1-\delta)$ for small $\delta (< \frac{1}{2})$. However, as ε decreases so that $\varepsilon' < \delta$, this equilibrium (S, S, C) cannot be an ε' -PBE because the consistency condition does not allow such behavior as far as $f(C|S) = \delta > \varepsilon'$. Therefore, (S, S, C) , which shows the first-mover advantage, is an ε -PBE for $\varepsilon > \delta$ but not a limit PBE.

Can the static Nash outcome (C, C) be supported as an ε -PBE? Player II chooses C after receiving a signal C (at information set h_2). After receiving S , however, player II doubts the possibility that player I selected the equilibrium action C because the probability that he receives S is too small if player I chose C , i.e., $f(S|C) = \delta < \varepsilon$. In this case, arbitrary belief is allowed. If he believes that player I chose C , the optimal move for him to choose C . Therefore, player I will opt for C . This scenario implies that the static Nash outcome is also an ε -PBE. Furthermore, for $\varepsilon' < \delta$, ε' -likelihood consistency allows him only to believe that player I chose C at information set h_2 , thus implying that player I will select C . Therefore, (C, C, C) is a limit PBE. In fact, it directly follows from Proposition 1 because (C, C, C) is a PBE.

We can generalize this result to games with any finite actions. Thus, $n = |A_1|$.

²⁸ Player II knows that player I is supposed to choose S in equilibrium. Therefore, in this case, the equilibrium meaning of player I's behavior is S regardless of what signal player II observes.

For any $a_1 \in A_1$, player II is assumed to observe $y = a_1$ with probability $1 - \delta$ and receive a wrong signal $y = a' (\neq a_1)$ with equal probabilities $\frac{\delta}{n-1}$. $\pi_i(a_1, a_2)$ is the payoff of player i , (a_1^N, a_2^N) is the Nash equilibrium in the static game without noise, and (a_1^s, a_2^s) is the Stackelberg outcome in the sequential game without noise. Additionally, $\pi_1^N \equiv \pi_1(a_1^N, a_2^N)$ and $\pi_1^s \equiv \pi_1(a_1^s, a_2^s)$. We define $\underline{a}_1 = \arg \min \pi_1(a_1, a_2^{BR}(a_1))$ by the belief that gives player I the minimum payoff when player II responds optimally along the best response function. We call \underline{a}_1 the most pessimistic belief and $a_2^{BR}(\underline{a}_1)$ the most severe threat. We also define the payoff $\underline{\pi}_1 \equiv \max_{a_1} \pi_1(a_1, a_2^{BR}(\underline{a}_1))$ by the maximum payoff of player I when player II uses the most severe threat. Then, we have the following Folk Theorem.

Proposition 3 (Folk Theorem). *Define $A^* = \{a_1 \in A_1 \mid \pi_1(a_1, a_2^{BR}(a_1)) > \underline{\pi}_1\}$. Then, in a game with noise, for any $a_1 \in A^*$, $\bar{\delta} > 0$ exists such that for any δ such that $0 < \delta \leq \bar{\delta}$, $(a_1, a_2^{BR}(b(y)))$, where $b(a_1) = a_1$ and $b(y) = \underline{a}_1$ if $y \neq a_1$ is an ε -PBE for any $\varepsilon > 0$ such that $\delta < \varepsilon$. In particular, if $\pi_1^N > \underline{\pi}_1$, any a_1 between a_1^N and \underline{a}_1 is an ε -PBE action for such ε .*

The payoff $\underline{\pi}_1$ plays a similar role as the minmax payoff in the Folk Theorem of repeated games in that player I can obtain $\underline{\pi}_1$ at the most when player II penalizes her most severely by choosing $a_2^{BR}(\underline{a}_1)$. It is player I's payoff from her most profitable deviation when she expects the most severe threat. This scenario demonstrates that player I's actual hidden action is not necessarily identical to the belief \underline{a}_1 . We call $\underline{\pi}_1$ the minmax payoff of player I in a broad sense.

This Folk Theorem says that if the static Nash payoff is higher than the minmax payoff, any outcome (including the Stackelberg outcome) along the best response function of player II that yields her a higher payoff than the static Nash outcome can be supported as an ε -PBE for some small $\delta > 0$, as long as we pick $\varepsilon > \delta$. When a_1^* is supported in ε -PBE, the equilibrium payoff is not $\pi_1(a_1^*, a_2^{BR}(a_1^*))$ because $\pi_1(a_1^*, a_2^{BR}(\underline{a}_1))$ can be realized with a small probability depending on the observation y . Therefore, this Folk Theorem given in Proposition 3 is in terms of strategies rather than in terms of payoffs, unlike the Folk Theorem of repeated games.

We can also generalize the result to games with continuum actions. $(a_1^s, a_2^{BR}(a_1))$ is the Stackelberg outcome, which is not a static Nash equilibrium.

Proposition 4. *Assume that $|U^I(a_1, a_2)| \leq M$ for some $M \in (0, \infty)$. In a game with noise, $(a_1^s, a_2^{BR}(b))$ is an ε -PBE and a limit PBE if the following [UP] condition holds; $U^I(a_1^s, a_2^{BR}(b))$ is strictly monotonic in b and unbounded.*

This proposition has an important implication that the first-mover advantage, which was forged with even a slight noise in observing the choice of the first mover,

is resupported as an equilibrium outcome if we use ε -PBE or limit PBE as our equilibrium concept.

The sufficient condition provided in Proposition 4, which we call the unbounded penalty [UP] condition, is crucial to our result. Intuitively, if a value of y that is very unlikely given that a_1^ε is played is realized, player II must form a very extreme belief a_1^ε , which makes it possible for him to rationally respond (to a_1^ε) to lead to a very small $U^I(a_1, a_2^{BR}(a_1^\varepsilon))$ when the [UP] condition holds. This punishment strategy $a_2^{BR}(a_1^\varepsilon)$ can be called the “boiling-in-oil” strategy, as it is often referred to in the principal-agent literature.²⁹ Without the condition, player II’s best response even to such an extreme belief might have a mild effect in which the resulting utility is bounded below. Thus, a very harsh punishment by player II is not feasible.

Indeed, the [UP] condition is too strong. If this condition does not hold, the Stackelberg action a_1^ε may be supported as an ε -PBE for some $\varepsilon > 0$ but cannot be supported as a limit PBE. This outcome is consistent with the result in Bagwell’s example.

An interesting feature of this model is that player I knows that player II’s response does not depend on a_1 directly, i.e., her choice of a_1 does not affect player II’s response directly. Most of the time, player II behaves as a passive player without responding to a_1 . Notably, player II cannot respond to a_1 because he can observe y but not a_1 . Then, how can player I achieve the Stackelberg outcome? Player II can respond to a signal y that can be affected by a_1 . If player I deviates from the Stackelberg outcome, player II may respond by a punishment action that a value of y falling short of a threshold level triggers with a very small probability. This possibility prevents player I from deviating from the Stackelberg action.

We can ponder the implication of the credibility of a threat to punish in this model. If player I’s move is perfectly observed by player II, player II cannot successfully threaten to choose C regardless of player I’s choice because the threat is incredible. In other words, perfect observability drives player II to respond to his observation S , thereby opting for S . However, with even a slight noise in observation, player II can commit to choosing C . This threat is credible because player II can observe nothing. Hence, player II cannot achieve the Stackelberg outcome in Bagwell’s example and in the generalized model. Employing the concept of ε -PBE brings a similar effect to observability. If we employ ε -PBE, when player I chooses S in equilibrium, player II responds by selecting S if he observes a signal S with high probability. This decision is not because he observes her choosing S but because he infers that she will select S in equilibrium. Player II does not need to believe player I’s threat to choose S in the case that he observes a signal C because it is a very unlikely event given that player I actually selected S . In this case, he may respond by choosing C . Player II’s threat to choose

²⁹ See, for example, Rasmusen (1994).

C regardless of his observation is not credible even with imperfect observation. Therefore, commitment is impossible.

If a likelihood test in the case of noisy observation can have a similar effect of perfect observability, what is the essential difference that distinguishes the case of noisy observation from the case of perfect observability? Proposition 3 says that if $\pi_1^N > \underline{\pi}_1$, any a_1^* between a_1^N and a_1^s can be an ε -PBE action in the case of noisy observation. However, it is usually not an equilibrium action in the case of perfect observation. If a_1 is perfectly observable, player II responds by $a_2^{BR}(a_1)$. Therefore, if player I deviates from a_1^* to a_1^s , it induces player II to respond from $a_2^{BR}(a_1^*)$ to $a_2^{BR}(a_1^s)$, which puts player I at an advantage. This result implies that $a_1^* \neq a_1^s$ cannot be an equilibrium outcome. However, if a_1 is imperfectly observed with noise, deviating from a_1^* to a_1^s does not induce player II to respond to $a_2^{BR}(a_1^s)$ but to $a_2^{BR}(\underline{a}_1)$ with some probability because such a deviation leads to the belief of either $b(y) = a_1^*$ or \underline{a}_1 , not $b(y) = a_1^s$. Given the possibility of the pessimistic belief $b(y) = \underline{a}_1$, player I may not deviate from the proposed equilibrium action a_1^* , even if $a_1^* \neq a_1^s$.

IV. Application: Price Competition Model

We can apply our equilibrium concepts to various real situations. We consider the following specific model, which will be helpful to obtain equilibrium strategies and beliefs explicitly.

Two firms that produce differentiated substitutes compete against each other by choosing prices. They face symmetric demand functions. The demand function for the good produced by firm i is given by $q_i = \alpha - \beta p_i + \gamma p_j$ for $j \neq i$, $i = 1, 2$, where $\alpha, \beta, \gamma > 0$ and $2\beta > \gamma$. For simplicity, we assume that the marginal cost is zero. Then, we can compute the profit function of firm i as

$$\pi^i(p_i, p_j) = p_i(\alpha - \beta p_i + \gamma p_j), \quad (2)$$

and the best response function as

$$p_i^{BR} = \frac{\alpha + \gamma p_j}{2\beta}. \quad (3)$$

p_1 and p_2 are strategic complements because the $p_i^{BR}(p_j)$ function has a positive slope.³⁰

³⁰ If the players compete in strategic complements such as prices, the second-mover advantage

If the firms cannot observe the price of each other, they end up with the static Nash prices $(p_1^N, p_2^N) = (\frac{\alpha}{2\beta-\gamma}, \frac{\alpha}{2\beta-\gamma})$. On the other hand, if firm 1 first chooses its price as a Stackelberg leader and the price is perfectly observed by firm 2, we can expect the well-known Stackelberg outcome (p_1^s, p_2^s) to be realized in a subgame perfect equilibrium. Given that firm 1 expects firm 2 to respond optimally according to its best response function, $p_2^{BR}(p_1) = \frac{\alpha + \gamma p_1}{2\beta}$, firm 1 will choose p_1 to

$$\max_{p_1} \pi^I(p_1, p_2^{BR}(p_1)) = p_1 \left[\alpha - \beta p_1 + \frac{\gamma(\alpha + \gamma p_1)}{2\beta} \right],$$

Thus leading to equilibrium prices

$$p_1^s = \frac{\beta + \frac{\gamma}{2}}{2\beta^2 - \gamma^2} \alpha, \quad (4)$$

$$p_2^s = \frac{\alpha + \gamma p_1}{2\beta} = \frac{\beta + \frac{\gamma}{2} - \frac{\gamma^2}{4\beta}}{2\beta^2 - \gamma^2} \alpha, \quad (5)$$

and the resulting profit of firm 1

$$\pi^I(p_1^s, p_2^s) = \frac{\alpha^2(2\beta + \gamma)^2}{8\beta(2\beta^2 - \gamma^2)}. \quad (6)$$

If firm 2 cannot observe p_1 perfectly but only with some noise, we can show that the Stackelberg leader's price p_1^s cannot be a PBE outcome with even a slight noise, insofar as the noise has full support $(-\infty, \infty)$. Given that every signal (observation) y is possible in equilibrium, firm 2 must respond to any observation y by $p_2^{BR}(b(y))$, which is just the same as $p_2^{BR}(p_1^s)$ because $b(y) = p_1^s$ in equilibrium for all y . Knowing that firm 2's response will not be affected by firm 1's choice, firm 1 will deviate from p_1^s by slightly cutting its price, which overturns the Stackelberg equilibrium.

We will now resort to alternative equilibrium concepts, ε -PBE and limit PBE. To characterize the equilibrium strategies explicitly, we assume that after firm 1 chooses its price p_1 , firm 2 observes a signal $y = p_1 + \eta$, where the noise η is normally distributed with mean zero and variance σ^2 , i.e., $\eta \sim N(0, \sigma^2)$. Therefore, the density function of y given p_1 is

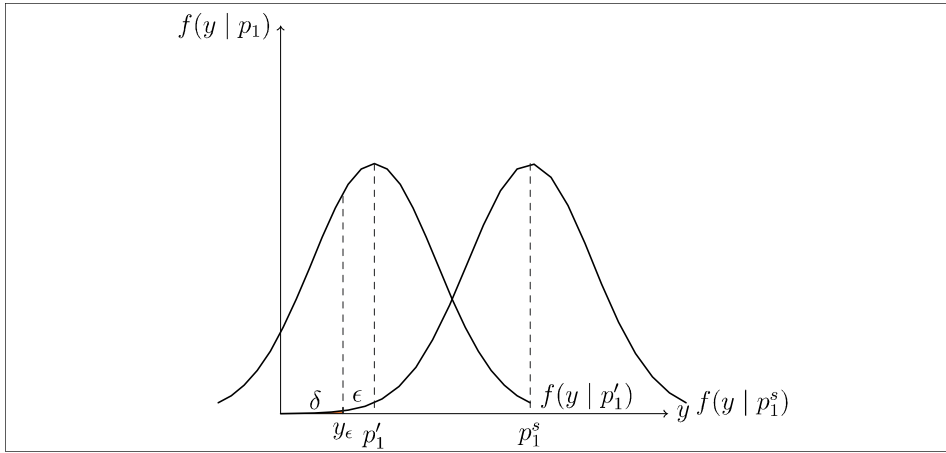
rather than the first-mover advantage appears in the sense that the equilibrium profit of the second mover is higher than the equilibrium profit of the first mover. See, for example, Gal-Or (1985). However, throughout this section, we maintain the term of the first-mover advantage simply because the first mover's profit is greater than in the static Nash equilibrium.

$$f(y | p_1) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-p_1}{\sigma}\right)^2}. \quad (7)$$

For a given ε , we can determine the corresponding cutoff value of the observation y_ε for the left tail event by using the conditional density function:

$$L(p_1^s; y_\varepsilon) = f(y_\varepsilon | p_1^s) = \varepsilon. \quad (8)$$

[Figure 3] Shift in the Conditional Density Function ($p_1^s > p_1'$)



A cutoff value has the following meaning and interpretation: if a lower signal than the cutoff value y_ε is observed, firm 2 believes that firm 1 did not select the equilibrium price p_1^s on the grounds that the likelihood $L(p_1^s; y)$ is less than ε for any $y \leq y_\varepsilon$. (See Figure 3.) Given that $y \sim N(p_1^s, \sigma)$, y_ε is determined from the following equilibrium:

$$f(y_\varepsilon | p_1^s) \equiv \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{1}{2} \left(\frac{y_\varepsilon - p_1^s}{\sigma} \right)^2 \right] = \varepsilon, \quad (9)$$

i.e.,

$$\left(\frac{y_\varepsilon - p_1^s}{\sigma} \right) = -2 \ln(\sqrt{2\pi}\sigma\varepsilon) > 0. \quad (10)$$

Therefore, for any given $\varepsilon > 0$, the confidence interval for y (equilibrium path) is determined by $p_1^s - \rho_\varepsilon < y < p_1^s + \rho_\varepsilon$, where $\rho_\varepsilon = -\sigma \ln(\sqrt{2\pi}\sigma\varepsilon)$ is the

maximum permissible error. If $y \leq p_1^s - \rho_\varepsilon$ or $y \geq p_1^s + \rho_\varepsilon$, we regard it as an almost off-the-equilibrium event. The cutoff value of y is $y_\varepsilon = p_1^s - \rho_\varepsilon$, which determines the left tail event.

We now consider the incentive compatibility condition of firm 1. The expected profit of firm 1 can be computed as follows:

$$E[\pi^I] = \int_{-\infty}^{y_\varepsilon} \pi^I(p_1, p_2^\varepsilon) f(y | p_1) dy + \int_{y_\varepsilon}^{\infty} \pi^I(p_1, p_2^s) f(y | p_1) dy, \quad (11)$$

where $p_2^s = p_2^{BR}(p_1^s)$ and p_2^ε denote a threat price that will be applied in case a signal y such that $y \leq y_\varepsilon$ is observed. In addition, $\delta \equiv \int_{-\infty}^{y_\varepsilon} f(y | p_1^s) dy = \mathbb{P}(y \leq y_\varepsilon | p_1^s)$.³¹ Then, the first-order condition characterizing the incentive compatibility condition requires

$$\begin{aligned} \left. \frac{\partial E(\pi^I)}{\partial p_1} \right|_{p_1=p_1^s} &= \pi_1^I(p_1^s, p_2^\varepsilon) \delta + \pi^I(p_1^s, p_2^\varepsilon) \int_{-\infty}^{y_\varepsilon} f_{p_1}(y | p_1^s) dy \\ &\quad + \pi_1^I(p_1^s, p_2^s) (1 - \delta) + \pi^I(p_1^s, p_2^s) \int_{y_\varepsilon}^{\infty} f_{p_1}(y | p_1^s) dy \\ &= 0, \end{aligned} \quad (12)$$

where $\pi^I(p_1, p_2) = p_1(\alpha - \beta p_1 + \gamma p_2)$ and $\pi_1^I(p_1, p_2) = \alpha + \gamma p_2 - 2\beta p_1$.³² Intuitively, if firm 1 increases its price, it reduces its profit directly but also decreases the probability that the observed price falls into the left tail event, thereby triggering a punishment. Therefore, p_1^s balances the marginal loss in the profit with the marginal benefit from a lower expected punishment.

By using Leibniz's rule,

$$\int_a^b \frac{\partial}{\partial x} f(x; t) dt = \frac{d}{dx} \left(\int_a^b f(x, t) dt \right), \quad (13)$$

we have

$$\begin{aligned} &\pi^I(p_1^s, p_2^\varepsilon) \int_{-\infty}^{y_\varepsilon} f_{p_1}(y | p_1^s) dy + \pi^I(p_1^s, p_2^s) \int_{y_\varepsilon}^{\infty} f_{p_1}(y | p_1^s) dy \\ &= p_1^s(\alpha - \beta p_1^s + \gamma p_2^s) \int_{-\infty}^{y_\varepsilon} f_{p_1}(y | p_1^s) dy + \gamma p_1^s(p_2^\varepsilon - p_2^s) \int_{-\infty}^{y_\varepsilon} f_{p_1}(y | p_1^s) dy \\ &= \gamma p_1^s(p_2^\varepsilon - p_2^s) \int_{-\infty}^{y_\varepsilon} f_{p_1}(y | p_1^s) dy \end{aligned}$$

³¹ This equation corresponds with the p value in hypothesis testing.

³² In the Appendix, we prove that the second-order condition is satisfied.

given that

$$\int_{-\infty}^{\infty} f_{p_1}(y | p_1^s) dy = \frac{d}{dp_1} \left(\int_{-\infty}^{\infty} f(y | p_1^s) dy \right) = 0.$$

Additionally, we have

$$\int_{-\infty}^{y_\varepsilon} f_{p_1}(y | p_1^s) dy = \frac{d}{dp_1} \int_{-\infty}^{y_\varepsilon} f(y | p_1^s) dy = \frac{dF(y_\varepsilon | p_1)}{dp_1} \bigg|_{p_1=p_1^s}. \quad (14)$$

We know that the normal distribution function is

$$F(y_\varepsilon | p_1) = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{y_\varepsilon - p_1}{\sqrt{2}\sigma} \right) \right],$$

where $\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-t^2} dt$. This function implies that

$$\begin{aligned} \frac{dF(y_\varepsilon | p_1)}{dp_1} \bigg|_{p_1=p_1^s} &= \frac{1}{\sqrt{\pi}} e^{-\left(\frac{y_\varepsilon - p_1^s}{\sqrt{2}\sigma}\right)^2} \cdot \left(-\frac{1}{\sqrt{2}\sigma} \right) \\ &= -\frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{y_\varepsilon - p_1^s}{\sqrt{2}\sigma}\right)^2} \\ &= -L(p_1^s; y_\varepsilon) \\ &= -\varepsilon. \end{aligned} \quad (15)$$

Thus, equation (12) is reduced to

$$\pi_1^I(p_1^s, p_2^s) = \gamma(p_2^\varepsilon - p_2^s)(\delta + \varepsilon p_1^s). \quad (16)$$

The left-hand side is the marginal loss in the profit from increasing the price, and the right-hand side is its marginal gain from a fall in the expected punishment. This outcome leads to

$$p_2^\varepsilon = p_2^s + \frac{\pi_1^I(p_1^s, p_2^s)}{\gamma(\delta + \varepsilon p_1^s)}. \quad (17)$$

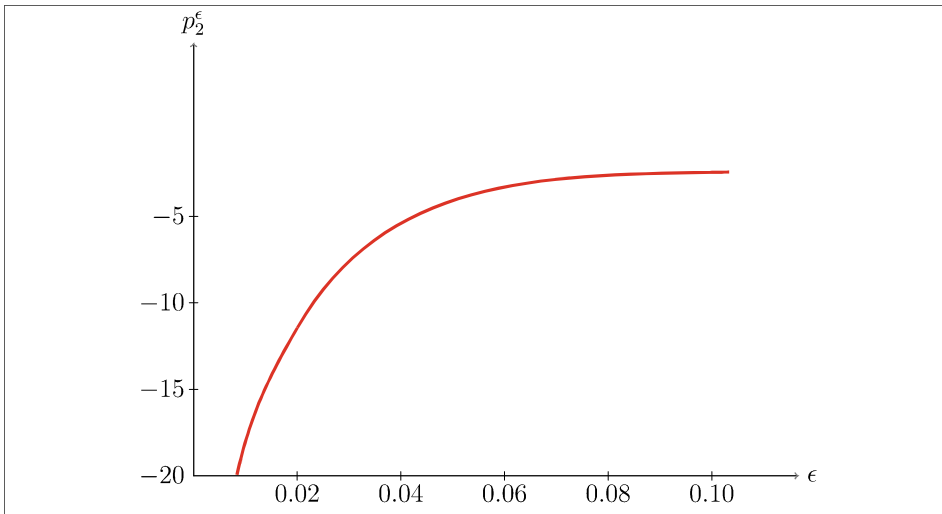
Given that $\pi_1^I(p_1^s, p_2^s) < 0$, $p_2^\varepsilon < p_2^s$. Additionally, if $\varepsilon, \delta \rightarrow 0$, then $p_2^\varepsilon \rightarrow -\infty$. Is p_2^ε a credible threat? It is an optimal response to some belief $b = p_1^\varepsilon$ such that

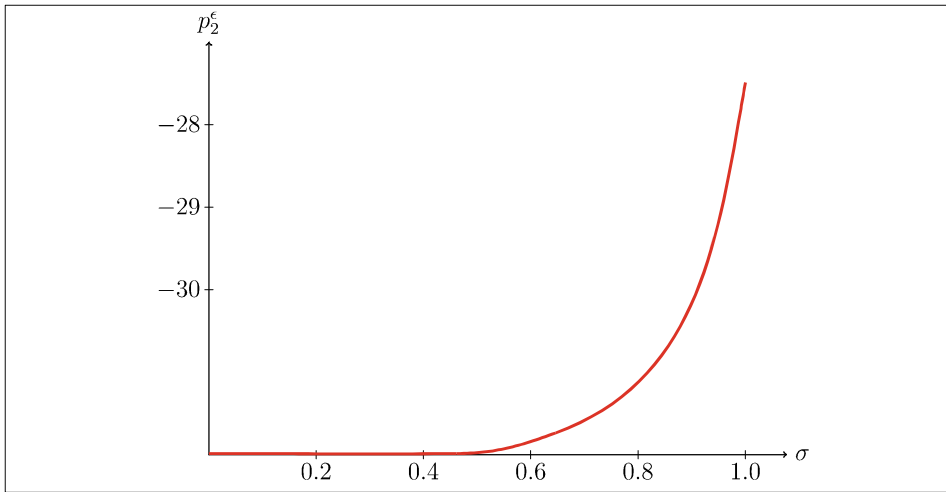
$p_2^\epsilon = p_2^{BR}(p_1^\epsilon)$. Given that $p_2^{BR}(p_1)$ is surjective, p_1^ϵ , which is a threat belief, always exists. A threat to p_2^ϵ is credible because it is the best response amid threat belief that firm 1 chose a very low price p_1^ϵ . Such a low price threat can frustrate firm 1's incentive to cut its price secretly. Given that equation (17) has a solution for any $\epsilon > 0$, the Stackelberg outcome is an ϵ -PBE for any $\epsilon > 0$, thus resulting in a limit PBE. In this specific model, the [UP] condition holds, i.e., $\lim_{p_1^\epsilon \rightarrow -\infty} \pi(p_1^\epsilon, p_2^{BR}(p_1^\epsilon)) = \lim_{p_2^\epsilon \rightarrow -\infty} \pi(p_1^\epsilon, p_2^\epsilon) = -\infty$.

Notably, the Stackelberg outcome cannot be a strong limit PBE because neither $\lim_{\epsilon \rightarrow 0} p_1^\epsilon$ nor $\lim_{\epsilon \rightarrow 0} p_2^\epsilon$ exists. This result implies that the Stackelberg leader's price cannot be a PBE outcome either, as in Proposition 2.

Figure 4 illustrates the equilibrium values of firm 2's threat price, p_2^ϵ , with a change in ϵ for the parameter values of $\alpha = 1$, $\beta = 3$, $\gamma = 2$, and $\sigma = 1$. As ϵ decreases, the threat price declines rapidly. This monotonicity reflects the intuition that a smaller probability of a tail event due to a lower ϵ must be accompanied by a harsher punishment to maintain the same deterrence power. The same intuition can be applied to a change in the variance of noise. Figure 5 shows a monotonic decrease in the threat price as σ^2 decreases for the value of $\epsilon = 0.1$. Again, a decrease in firm 1's utility due to an excessively low price by firm 2 must be compensated for a smaller probability of a tail event by more precise information.

[Figure 4] Equilibrium Values for p_2^ϵ with a Change in ϵ



[Figure 5] Equilibrium Values for p_2^ε with a Change in σ 

V. Conclusion

We introduced two equilibrium concepts, namely, ε -PBE and limit PBE, which slightly weaken the consistency requirement of PBE. We demonstrated that the first-mover advantage that disappeared in those games can be fully recovered by invoking these equilibrium concepts. These concepts can be useful in dynamic games with noisy signals that have unbounded support. They can also be used in cheap-talk games in which every message is possible in equilibrium because the receiver can punish the sender when observing a message that is very unlikely in equilibrium albeit possible.³³

However, we admit that our result of the existence of a limit PBE depends crucially on the [UP] condition, which makes the “boiling-in-oil” strategy feasible. This strategy is optimal given the extremely pessimistic belief. However, the punisher can also be severely penalized by the belief. We assert that it is indeed a strong sufficient condition but not necessary for the existence of a limit PBE. Thus, a milder condition can guarantee its existence. We hope to see additional developments in those equilibrium concepts in the near future.

³³ See, for example, Jung and Kim (2019) and Kim (2019).

Appendix

Proof of Proposition 1: (i) If a_1^* is a limit PBE, for any $\varepsilon > 0$ such that $\varepsilon < \bar{\varepsilon}$ for some $\bar{\varepsilon} > 0$, $b(y; \varepsilon)$ exists such that $(a_1^*, a_2^*(b), b(y; \varepsilon))$ is an ε -PBE. Therefore, the proof is done.

(ii) If $(a_1^*, a_2^*(b), b(y))$ is a PBE, for any $\varepsilon > 0$, one can take $b(y; \varepsilon)$ satisfying ε -PBE. For any $\varepsilon > 0$, take $b(y; \varepsilon) = b(y)$. Then, $(a_1^*, a_2^*(b), b(y; \varepsilon)) = (a_1^*, a_2^*(b), b(y))$ is an ε -PBE because $b(y; \varepsilon) = b(y)$ satisfies consistency, thus implying ε -likelihood consistency. Given that this outcome holds for any $\varepsilon > 0$, a_1^* is a limit PBE. ■

Proof of Proposition 2: (i) (\Leftarrow) The proof is similar to the proof of Proposition 1(ii). If $(a_1^*, a_2^*(b), b(y))$ is a PBE, for any ε_n such that $\varepsilon_n \rightarrow 0$, take $b(y; \varepsilon_n) = b(y)$. Then, by Proposition 1, $(a_1^*, a_2^*(y; \varepsilon_n), b(y; \varepsilon_n)) = (a_1^*, a_2^*(b), b(y))$ is ε_n -PBE. Given that $\lim_{n \rightarrow \infty} b(y; \varepsilon_n) = b(y)$, it is a strong limit PBE.

(ii) (\Rightarrow) To simplify the proof, we assume that $\text{supp}(f | a_1) = Y$ for any $a_1 \in A_1$. If $(a_1^*, a_2^*(b), b(y))$ is a strong limit PBE, for any $\varepsilon (> 0)$, $b(y; \varepsilon)$ exists such that $\lim_{\varepsilon \rightarrow 0} b(y; \varepsilon) = b(y)$ and $a_2^*(b)$ is a BR to $b(y; \varepsilon)$, i.e.,

$$U^H(b(y; \varepsilon), a_2^*(b(y; \varepsilon))) \geq U^H(b(y; \varepsilon), a_2), \forall a_2 \in A_2. \quad (18)$$

Given that this outcome holds for any $\varepsilon > 0$, we take limits to obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} U^H(b(y; \varepsilon), a_2^*(b(y; \varepsilon))) &= U^H(\lim_{\varepsilon \rightarrow 0} b(y; \varepsilon), \lim_{\varepsilon \rightarrow 0} a_2^*(b(y; \varepsilon))) \quad (\text{by continuity of } U^H) \\ &= U^H(\lim_{\varepsilon \rightarrow 0} b(y; \varepsilon), a_2^*(\lim_{\varepsilon \rightarrow 0} b(y; \varepsilon))), \quad (\text{by continuity of } a_2^*(b) \text{ in } b) \\ &= U^H(b(y), a_2^*(b(y))) \quad (\because \lim_{\varepsilon \rightarrow 0} b(y; \varepsilon) = b(y)) \\ &\geq U^H(b(y), a_2), \forall a_2 \in A_2. \quad (\text{by Inequality (18)}) \end{aligned}$$

$(a_2^*(b))$ is continuous in b because $a_2^*(\cdot) = a_2^{BR}(\cdot)$. Additionally, by the definition of limit PBE, we have

$$\int_Y U^I(a_1^*, a_2^*(b(y; \varepsilon))) f(y | a_1^*) dy \geq \int_Y U^I(a_1, a_2^*(b(y; \varepsilon))) f(y | a_1) dy, \forall a_1 \in A_1.$$

Taking limits to both sides, we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_Y U^I(a_1^*, a_2^*(b(y; \varepsilon))) f(y | a_1^*) dy &= \int_Y U^I(a_1^*, a_2^*(b(y))) f(y | a_1^*) dy \\ &\geq \lim_{\varepsilon \rightarrow 0} \int_Y U^I(a_1, a_2^*(b(y; \varepsilon))) f(y | a_1) dy = \int_Y U^I(a_1, a_2^*(b(y))) f(y | a_1) dy, \end{aligned}$$

for all $a_1 \in A_1$ given that $\lim_{\varepsilon \rightarrow 0} b(y; \varepsilon) = b(y)$ by the definition of the strong limit PBE. Hence, $(a_1^*, a_2^*(b(y)))$ is a PBE. ■

Proof of Proposition 3: For any $\varepsilon > \delta$, $a_1^* \in A^*$ is player I's action of ε -PBE. By suppressing the subscript of player I in the payoff function, we can compute the equilibrium payoff of player I as

$$\Pi^* \equiv (1 - \delta)\pi^* + \delta\underline{\pi}(a_1^*),$$

where $\pi^* = \pi(a_1^*, a_2^{BR}(a_1^*))$ because the ε -likelihood consistency allows $b(y) = \underline{a}_1$ if $y \neq a_1^*$ with probability δ . Her payoff from the most profitable deviation is

$$\max_{a_1' \neq a_1^*} \Pi \equiv \left[\left(1 - \frac{\delta}{n-1} \right) \pi(a_1', a_2^{BR}(\underline{a}_1)) + \frac{\delta}{n-1} \pi(a_1', a_2^{BR}(a_1^*)) \right],$$

because player II observes $y = a_1^*$ with probability $\frac{\delta}{n-1}$ if $a_1 \neq a_1^*$.

We have $\pi^N > \underline{\pi}$ because $a_1^* \in A^*$. Therefore, $\pi^* > \pi(a_1', a_2(\underline{a}_1))$ for any $a_1' \neq a_1^*$. This outcome implies that a small value of $\delta > 0$ exists such that $\Pi^* > \Pi$. Take such δ and denote it by $\bar{\delta}$. Then, for any $\delta \leq \bar{\delta}$, $(a_1^*, a_2^{BR}(b(y)))$ is an ε -PBE because $\varepsilon > \delta$, where $b(a_1^*) = a_1^*$ and $b(y) = \underline{a}_1$ if $y \neq a_1^*$. ■

Proof of Proposition 4: Without loss of generality, assume that $\lim_{b \rightarrow -\infty} U^I(a_1^s, a_2^{BR}(b)) = -\infty$. Given that $(a_1^s, a_2^{BR}(a_2^s))$ is not a static Nash equilibrium, player I has an incentive to deviate to either $a_1 < a_1^s$ or $a_1 > a_1^s$. Without loss of generality, we only consider the case that $a_1 < a_1^s$.

Lemma 1. $a_2^{BR}(a_1)$ is surjective.

Proof. It is sufficient to show that $a_2^{BR}(a_1)$ is not bounded by K for some $K \in (0, \infty)$, i.e., $|a_2^{BR}(a_1)| \leq K$. Suppose $a_2^{BR}(a_1)$ is bounded. By the [UP] condition, $\lim_{b \rightarrow -\infty} U^I(a_1^s, a_2^{BR}(b)) = -\infty$. Then, by continuity of U^I , we have $\lim_{b \rightarrow -\infty} U^I(a_1^s, a_2^{BR}(b)) = U^I(a_1, \lim_{b \rightarrow -\infty} a_2^{BR}(b)) = -\infty$. This scenario is not possible because $U^I(a_1, a_2)$ is continuous for all $a_2 \in \mathbb{R}$ and $a_2^{BR}(b)$ is bounded. Therefore, $a_2^{BR}(a_1)$ must be not bounded. The case in which $\lim_{b \rightarrow \infty} U^I(a_1^s, a_2^{BR}(b)) = -\infty$ is similar. Thus, we omit the proof for this case. ■

For any fixed $\varepsilon > 0$, take y_ε such that $L(a_1^s, y_\varepsilon) = f(y_\varepsilon | a_1^s) = \varepsilon$, where $y_\varepsilon < a_1^s$. Then, ε -PBE allows player II to punish player I if he observes $y < y_\varepsilon$ by assigning the posterior belief $b(a_1 | y) = a_1^\varepsilon < a_1^s$ and otherwise (if he observes

$y \geq y_\varepsilon$) $b(a_1 | y) = a_1^\varepsilon$. Therefore, $\delta \equiv \int_{-\infty}^{y_\varepsilon} f(y | a_1^\varepsilon) dy = \mathbb{P}(y \leq y_\varepsilon | a_1^\varepsilon)$.

Suppressing the superscript I of U^I , we can compute player I's expected utility as

$$E[U] = U(a_1, a_2^{BR}(a_1^\varepsilon)) \int_{-\infty}^{y_\varepsilon} f(y | a_1) dy + U(a_1, a_2^{BR}(a_1^\varepsilon)) \int_{y_\varepsilon}^{\infty} f(y | a_1) dy. \quad (19)$$

The first-order condition requires

$$\left. \frac{\partial E[U]}{\partial a_1} \right|_{a_1=a_1^\varepsilon} = \varphi_1 + \varphi_2 = 0, \quad (20)$$

where

$$\begin{aligned} \varphi_1 &= U_1(a_1^\varepsilon, a_2^{BR}(a_1^\varepsilon)) \int_{-\infty}^{y_\varepsilon} f(y | a_1^\varepsilon) dy + U_1(a_1^\varepsilon, a_2^{BR}(a_1^\varepsilon)) \int_{y_\varepsilon}^{\infty} f(y | a_1^\varepsilon) dy, \\ \varphi_2 &= U(a_1^\varepsilon, a_2^{BR}(a_1^\varepsilon)) \int_{-\infty}^{y_\varepsilon} f_{a_1}(y | a_1^\varepsilon) dy + U(a_1^\varepsilon, a_2^{BR}(a_1^\varepsilon)) \int_{y_\varepsilon}^{\infty} f_{a_1}(y | a_1^\varepsilon) dy. \end{aligned}$$

By using $a_2^\varepsilon = a_2^{BR}(a_1^\varepsilon)$, $a_2^\varepsilon = a_2^{BR}(a_1^\varepsilon)$, $\int_{-\infty}^{y_\varepsilon} f(y | a_1^\varepsilon) dy = \delta$ and $\int_{-\infty}^{y_\varepsilon} f_{a_1}(y | a_1^\varepsilon) dy = -\varepsilon$ from (14) and (15), we can simplify equation (20) into

$$\varepsilon(U(a_1^\varepsilon, a_2^\varepsilon) - U(a_1^\varepsilon, a_2^\varepsilon)) = \delta(U_1(a_1^\varepsilon, a_2^\varepsilon) - U_1(a_1^\varepsilon, a_2^\varepsilon)) - U_1(a_1^\varepsilon, a_2^\varepsilon). \quad (21)$$

The left-hand side (LHS) of (21) is a reduction in the penalty probability (ε) due to a marginal increase in a_1 times the magnitude of the penalty ($U(a_1^\varepsilon, a_2^\varepsilon) - U(a_1^\varepsilon, a_2^\varepsilon)$), which is a reduction in the expected cost from a marginal increase in a_1 . The right-hand side (RHS) of (21) is a reduction in the expected gain due to a marginal deviation from a_1^ε . It is straightforward to see that (22) follows from (21).

$$U(a_1^\varepsilon, a_2^\varepsilon) - U(a_1^\varepsilon, a_2^\varepsilon) = \frac{\delta}{\varepsilon} (U_1(a_1^\varepsilon, a_2^\varepsilon) - U_1(a_1^\varepsilon, a_2^\varepsilon)) - \frac{U_1(a_1^\varepsilon, a_2^\varepsilon)}{\varepsilon}. \quad (22)$$

Notably,

$$\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\varepsilon} = \lim_{y_\varepsilon \rightarrow -\infty} \frac{\int_{-\infty}^{y_\varepsilon} f(y | a_1^\varepsilon) dy}{f(y_\varepsilon | a_1^\varepsilon)}$$

$$\begin{aligned}
&= \lim_{z \rightarrow -\infty} \frac{\int_{-\infty}^{y_\varepsilon} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}{\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}} \quad (\because \text{by letting } z = \frac{y_\varepsilon - a_1^\varepsilon}{\sigma}) \\
&= \lim_{z \rightarrow -\infty} \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}}{-\frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}} \quad (\because \text{by L'Hospital's rule}) \\
&= \lim_{z \rightarrow -\infty} -\frac{1}{z} \\
&= 0.
\end{aligned} \tag{23}$$

Given that $\frac{\delta}{\varepsilon} \rightarrow 0$, $|U_1(a_1^\varepsilon, a_2^\varepsilon)| \leq M$ for some $M > 0$, and $\frac{U_1(a_1^\varepsilon, a_2^\varepsilon)}{\varepsilon}$ is finite for any $\varepsilon > 0$, the RHS of (22) is bounded. Therefore, $a_2^\varepsilon \in \mathbb{R}$ satisfies (21) by the intermediate value theorem due to the continuity of U with respect to a_2 , the [UP] condition and Lemma 1.

Given that $a_2^{BR}(a_1)$ is surjective by Lemma 1, we can define $a_1^\varepsilon \in (-\infty, \infty)$ such that $a_2^\varepsilon = a_2^{BR}(a_1^\varepsilon)$ for any $a_2^\varepsilon \in \mathbb{R}$. Then, for any $\varepsilon > 0$, $(a_1^\varepsilon, a_2^{BR}(a_1^\varepsilon))$ is an ε -PBE where

$$\begin{aligned}
b(y; \varepsilon) &= \begin{cases} a_1^\varepsilon & \text{if } L(a_1^\varepsilon; y) > \varepsilon \\ a_1^\varepsilon & \text{if } L(a_1^\varepsilon; y) \leq \varepsilon, \end{cases} \\
a_2^{BR}(b(y; \varepsilon)) &= \begin{cases} a_2^\varepsilon & \text{if } L(a_1^\varepsilon; y) > \varepsilon \\ a_2^\varepsilon & \text{if } L(a_1^\varepsilon; y) \leq \varepsilon, \end{cases}
\end{aligned}$$

Given that a_1^ε and a_2^ε exist for any ε , a_1^ε is a limit PBE. ■

Lemma 2. *In the price competition model, the Stackelberg price p_1^* satisfies the second-order condition of optimization, i.e., $\left. \frac{\partial^2 E(\pi^I)}{\partial p_1^2} \right|_{p_1 = p_1^*} < 0$.*

Proof. The second-order condition requires

$$\frac{\partial^2 E(\pi^I)}{\partial p_1^2} = \psi_1 + \psi_2 + \psi_3,$$

where

$$\psi_1 = \pi_{11}(p_1^\varepsilon, p_2^\varepsilon) \int_{-\infty}^{y_\varepsilon} f(y | p_1) dy + \pi_{11}(p_1^\varepsilon, p_2^\varepsilon) \int_{y_\varepsilon}^{\infty} f(y | p_1) dy,$$

$$\begin{aligned}\psi_2 &= 2 \left[\pi_1(p_1^s, p_2^e) \int_{-\infty}^{y_e} \frac{\partial f(y | p_1)}{\partial p_1} dy + \pi_1(p_1^s, p_2^s) \int_{y_e}^{\infty} \frac{\partial f(y | p_1)}{\partial p_1} dy \right], \\ \psi_3 &= \pi(p_1^s, p_2^e) \int_{-\infty}^{y_e} \frac{\partial^2 f(y | p_1)}{\partial p_1^2} dy + \pi(p_1^s, p_2^s) \int_{y_e}^{\infty} \frac{\partial^2 f(y | p_1)}{\partial p_1^2} dy.\end{aligned}$$

Tedious calculations yield

$$\begin{aligned}\psi_1 &= -2\beta \int_{-\infty}^{\infty} f(y | p_1) dy = -2\beta < 0, \\ \psi_2 &= 2\gamma(p_2^e - p_2^s) \int_{-\infty}^{y_e} f_{p_1}(y | p_1) dy = -2\epsilon\gamma(p_2^e - p_2^s) < 0 \\ \psi_3 &= \pi(p_1^s, p_2^s) \int_{-\infty}^{\infty} \frac{\partial^2 f(y | p_1)}{\partial p_1^2} dy + \gamma p_1^s \int_{-\infty}^{y_e} \frac{\partial^2 f(y | p_1)}{\partial p_1^2} dy \\ &= \pi(p_1^s, p_2^s) \frac{\partial}{\partial p_1} \int_{-\infty}^{\infty} \frac{\partial f(y | p_1)}{\partial p_1} dy + \gamma p_1^s \frac{\partial}{\partial p_1} \int_{-\infty}^{y_e} \frac{\partial f(y | p_1)}{\partial p_1} dy.\end{aligned}$$

Given that $\int_{-\infty}^{\infty} \frac{\partial f(y | p_1)}{\partial p_1} dy = \int_{-\infty}^{\infty} f(y | p_1) \left(\frac{y - p_1}{\sigma}\right) dy = E[y - p_1 | p_1] / \sigma = 0$, ψ_3 is simplified to

$$\begin{aligned}\psi_3 &= \gamma p_1^s \frac{\partial}{\partial p_1} \int_{-\infty}^{y_e} f(y | p_1) \left(\frac{y - p_1}{\sigma}\right) dy \\ &= \gamma p_1^s \frac{\partial}{\partial p_1} \int_{-\infty}^{\frac{y_e - p_1^s}{\sigma}} z \phi(z) dz \\ &= \gamma p_1^s \frac{\partial}{\partial p_1} \mathbb{P} \left[z \leq \frac{y_e - p_1^s}{\sigma} \right] \\ &< 0,\end{aligned}$$

where z follows the standard normal distribution and $\phi(z)$ is the standard normal distribution that $z = \frac{y_e - p_1^s}{\sigma}$ follows. This outcome completes the proof. ■

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불완전한 신호를 갖는 동태적 게임에서의 새로운 균형개념의 제안*

김 정 유**

초 록 본 연구에서는 관찰의 오류로 인하여 균형에서 어떠한 신호도 가능한 동태적 게임을 고려한다. 이러한 불완전한 신호를 갖는 게임에서는 완전 베이지안 균형이나 순차적 균형이 요구하는 믿음의 일관성조건이 지나치게 강하여 타당성있는 결과를 배제하게 되므로 ε -완전 베이지안 균형이나 극한 완전균형과 같은 대체적 균형개념이 필요함을 주장한다. 위 두 개념은 균형전략이 선택되었다고 할때 신호의 우도가 ε 을 초과하는 경우에 만 믿음에 베이지안 룰을 적용한다는 다소 완화된 일관성조건에 의해 정의된다. 이러한 개념들은 희소한 사건에 대해서는 베이지안 룰로부터의 이탈경향이 보다 높다는 경험적 관찰과도 일치한다. 본 연구는 위 두 개념이 선도자의 행위를 관찰함에 아주 근소한 오류가 있더라도 소멸되는 선도자 이득을 특정 조건하에 다시 회복시켜 줄 수 있음을 보인다.

핵심 주제어: 불완전한 신호를 가진 동학게임, ε -우도 일관성, ε -완전 베이지안 균형, 극한 완전 베이지안 균형, 단순우도검정

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