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Efficiency and Revenue in Asymmetric Auctions*

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We study the efficiency and revenue properties of first- and second-price auctions in an environment where bidders' values are drawn from different binary distributions. We identify a necessary and sufficient condition for a first-price auction to induce an efficient allocation. The condition reveals the source of allocative inefficiencies in asymmetric first-price auctions. We further show that the seller's revenue is higher in a second-price auction than it is in a first-price auction whenever allocations in the two auction formats are efficient. We highlight how the difference in different bidder types' beliefs induces the results.

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I. Introduction

It is well known that when there is ex ante asymmetry among bidders (i.e., their values are drawn from different distributions), the allocation is often inefficient in first-price auctions and the revenue equivalence between first- and second-price auctions no longer holds. Although it is still a weakly dominant strategy for each bidder to bid his value in asymmetric second-price auctions, the case is often that no closed-form expression for bidding strategies in asymmetric first-price auctions exists. The consequent complications render the analysis of first-price auctions significantly more involved, which in turn limits our understanding of asymmetric auctions. In particular, the revenue properties of first- and second-price auctions,

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which are one of the most basic questions in auction theory, have been only investigated partially.

In this paper, we study allocative efficiency in first-price auctions and show that first- and second-price auctions yield different revenues, even if both allocations are efficient. As we shall discuss shortly, most studies in the asymmetric auction literature consider the case where allocations are not efficient in first-price auctions and attempt to find conditions under which one auction formation yields higher revenue than the other. We take an alternative approach. We restrict our attention to equilibrium with an efficient allocation and provide a necessary and sufficient condition for the existence of such an equilibrium. This reveals the source of allocative inefficiencies in asymmetric first-price auctions. Because the seller revenue is the same as the social surplus minus the bidders' payoffs and the social surplus is maximized with an efficient allocation, our approach helps us identify the underlying forces that cause the different revenues of the two auction formats after controlling for the social surplus.

To this end, we consider a simple model of asymmetric auctions. There are $N \geq 2$ bidder types that are ex-ante distinguishable and more than two bidders of any given type. Bidders draw their values from different binary distributions. In particular, each type-n bidder's value for the object is either 0 or v_n. Thus, each type is identified with the associated positive value v_n and the probability of drawing that value. Although the model is specific, it considers a rich environment with multiple bidder types and an arbitrary number of bidders. We also do not impose any stochastic relations among the distributions.

The two main technical results of this paper are as follows. First, we identify a necessary and sufficient condition for the first-price auction to induce an efficient allocation. Specifically, we show that if bidder types are ordered positively in terms of their associated positive values (i.e., $v_n < v_{n+1}$), then the allocation of the first-price auction is efficient if and only if type-n bidders have no incentive to outbid type-$n+1$ bidders. Second, provided that the resulting allocations are efficient in both first- and second-price auctions, we show that the seller's expected revenue is higher in the second-price auction than in the first-price auction.

Both results are driven by the fact that different types of bidders have different beliefs because of the ex-ante bidder asymmetries. In asymmetric auctions, bidders differ in two respects. First, each bidder's value is known only to the bidder, and thus, each bidder has a different value. We call this *difference-in-values*. Second, each bidder knows the value of one bidder of his own type (i.e., himself), whereas other type bidders do not. Consequently, bidders have different beliefs about the preferences of other bidders. We call this *difference-in-beliefs*. Although the first difference is present in any symmetric auction, the second difference arises only with asymmetries.

Given this observation, consider two bidders, one type n and one type n', and a type-n' bidder's deviation incentives to a type-n bidder's equilibrium bid. The difference-in-values means a bidder with a higher value bids a higher price; that is, a type-n' bidder has an incentive to outbid (resp., underbid) a type-n bidder if $v_{n'} > v_n$ (resp. $v_{n'} < v_n$). In contrast, the difference-in-beliefs always provides a type-n' bidder with an incentive to outbid type-n bidder (whether $v_{n'} > v_n$ or $v_{n'} < v_n$). This is because the former faces a lower probability of outbidding all type-n rivals than the latter, even if they bid the same price.[1] Consequently, a type-n' bidder is more willing to increase his bid than a type-n bidder is. Note that the two incentives work in the same direction if $v_{n'} > v_n$, but in the opposite direction if $v_{n'} < v_n$. As a result, a type-n' bidder has no incentive for downward deviation but may have an incentive for upward deviation.

These mechanisms imply the absence of unambiguous revenue ranking when the allocation is not efficient in first-price auctions. In that case, bidders have an incentive for upward deviation, that is, they are overly aggressive in the first-price auction, which positively affects the seller's revenue. Therefore, which auction format delivers a higher revenue to the seller depends on which force dominates. If the bidders' information rents resulting from the differences in their beliefs are sufficiently large, then the seller's revenue is lower in the first-price auction, whereas if the bidders have a sufficiently strong incentive to increase their bids, then the opposite is true. This explains the second result of the paper, that is, in the absence of any distributive distortions (i.e., in equilibrium with an efficient allocation), the seller's expected revenue is higher in the second-price auction than in the first-price auction.

Although there is a large body of literature on asymmetric auctions, most previous works have relied on specific assumptions on the number of bidders and/or valuation distributions because of technical difficulties related to first-price auctions. Maskin and Riley (1985) derive equilibrium bidding strategies of a first-price auction with binary distributions. In a later study, Maskin and Riley (2000) derive sufficient conditions for one auction format to be revenue-superior to the other between first- and second-price auctions. Plum (1992), Cheng (2006), and Kaplan and Zamir (2012) derive equilibrium strategies of first-price auctions, wherein the first two works consider cases of power distributions and the third considers asymmetric uniform distributions. Kirkegaard (2012) employs a mechanism design approach and provides sufficient conditions for the first-price auction to dominate the second-price auction. Doni and Menicucci (2013) assume binary distributions and show that the second-price auction yields a higher revenue than the first-price auction when the distribution functions satisfy certain conditions. Gavious and

[1] If the number of type-n bidders is m_n and the probability that a type-n bidder bids less than b is $G(b)$, then the probability of a type-n' bidder outbidding all type-n bidders is $G(b)^{m_n}$, whereas that of a type-n bidder is $G(b)^{m_n - 1}$.

Minchuk (2014) study an environment in which valuation distributions are "close to" uniform distributions and compare the revenues of the two auction formats. However, these works are all limited to auctions with only two bidders.

With an arbitrary number of bidders, Lebrun (1999) characterizes bidding strategies for first-price auctions, while Fibich, Gavious, and Sela (2004) show that the two auction formats are approximately revenue-equivalent when the asymmetry is "weak". However, these works assume that the bidders' valuation distributions have common support. Relaxing the last assumption, Lebrun (2006) provides the uniqueness of equilibrium for the first-price auction. Hubbard and Kirkegaard (2015) characterize equilibrium bidding strategies of first-price auctions with two bidder types and multiple bidders for each type. Kirkegaard (2016) compares the revenues of first- and second-price auctions based on the results established in Hubbard and Kirkegaard (2015).

Given the set up with binary distributions, our model is similar to that of Doni and Menicucci (2013) but differs from theirs and other studies in the literature in several ways. First, our model employs multiple bidder types and more than two bidders for each type.[2] While this approach makes it difficult to derive bidding strategies in first-price auctions, we provide an explicit characterization for an efficient equilibrium and also offer several examples of inefficient equilibria. Second, although we consider binary distributions for bidders' valuations, we allow for different supports and do not impose any stochastic relations on these distributions. Thus, our revenue ranking result does not depend on the details of valuation distributions, unlike in Doni and Menicucci (2013) and others.

The remainder of the paper is organized as follows. Section 2 introduces our auction environment. Section 3 contains our main results. Section 3.1 characterizes the equilibrium bidding strategies of second-price auctions. Section 3.2 constructs a unique bidding strategy profile for an efficient equilibrium in first-price auctions and presents a necessary and sufficient condition for the existence of such an equilibrium. Section 3.3 compares the revenues of first- and second-price auctions. Section 4 discusses the implications of a large number of bidders, continuous valuation distributions, and inefficient allocations in the first-price auction. Section 5 concludes the paper.

[2] If there are two bidders of different types, then our model includes those of Maskin and Riley (1985) and Doni and Menicucci (2013) (for the case $v_{1L} = v_{2L}$ in their model). If there are two bidder types with multiple bidders of each type, then our model can be viewed as a discrete version of those of Hubbard and Kirkegaard (2015) and Kirkegaard (2016).

II. Model

A seller wishes to sell an indivisible object. There are $N(\geq 2)$ types of bidders and the number of type-n bidders is given by $m_n(\geq 2)$. Each type-n bidder draws his value for the object independently and identically from a binary distribution. Specifically, a type-n bidder's value is $v_n(>0)$ with probability $p_n \in (0,1)$ and 0 with probability $1-p_n$. For notational simplicity, let $v_0=0$ and assume that $v_{n-1}<v_n$ for any $n=2,\dots,N$, without loss of generality. If a bidder whose realized value is v wins the object and pays b, then the bidder's payoff is $v-b$ and the seller's revenue is b. All other bidders obtain a zero payoff.

Three remarks are relevant to the environment. First, we use "type" to refer to a bidder's *ex-ante* characteristic (distribution function), not a bidder's *ex-post* realized value. Second, we do not impose any stochastic order on the valuation distributions. For instance, it can be the case that $p_n > p_{n+1}$, so that there is no clear first-order stochastic dominance. Finally, the assumption that each bidder has zero value with a positive probability is a convenient normalization and does not alter our results as long as each bidder's minimum value is the same.[3]

We define some notations for later use. Let $q_n := (1-p_n)^{m_n}$ and $Q_n := \prod_{k>n} q_k$ denote the probabilities that all type-n bidders and all bidders with types above n draws zero, respectively. Next, $r_n := (1-p_n)^{m_n-1}$ denotes the probability that m_n-1 out of m_n type-n bidders draw zero.

III. Equilibrium and Revenue

3.1. Second-Price Auction

We first analyze the second-price auction. As in the standard second-price auction with symmetric bidders, it is a weakly dominant strategy for each bidder to bid his true value, even with asymmetries. Hence, we focus on this case.

Let V_n^S denote the expected payoff of a type-n bidder who draws a positive value v_n. In equilibrium, he obtains a positive payoff if and only if no bidder of type-$n' \geq n$ draws a positive value, the probability of which is $Q_n r_n$.[4] The price he pays to the seller depends on the realizations of lower-type bidders' values. For any $n' < n$, the price is equal to $v_{n'}$ if no bidder between $n'+1$ and $n-1$ draws a

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[3] An alternative interpretation is that all type-n bidders put value v_n on the object, but each type-n bidder participates in the auction with probability p_n.

[4] The bidder wins with a positive probability, as long as no bidder whose type is *strictly* above n draws a positive value. If there is another type-n bidder with a positive value, then his payoff is zero, whether he wins or not.

positive value (with probability $\prod_{j=n'+1}^{n-1} q_j$), while at least one type-n' bidder does draw a positive value (with probability $1-q_{n'}$). Therefore, the price a type-n bidder pays, conditional on winning and obtaining a positive payoff, is

$$b_n^S := (1-q_{n-1})v_{n-1} + q_{n-1}(1-q_{n-2})v_{n-2} + \cdots + \prod_{j=2}^{n-1} q_j (1-q_1)v_1 + \prod_{j=2}^{n-1} q_j v_0 .$$

Proposition 1. *In the second-price auction, the expected payoff of a type-* n *bidder with value* v_n *is equal to* $V_n^S = Q_n r_n (v_n - b_n^S)$.

The seller's expected revenue can be calculated similarly. Although the exact solution is not necessary for the subsequent analysis, notably, the allocation is efficient in the second-price auction because a bidder with the highest value always wins.

3.2. First-Price Auction

Now we consider the first-price auction, focusing on the equilibrium in which the allocation is efficient. We refer to this equilibrium as an *efficient equilibrium*.[5] We first construct a unique equilibrium strategy profile that yields an efficient allocation and then characterize a necessary and sufficient condition for the existence of an efficient equilibrium. We restrict our attention to a *symmetric equilibrium* in which bidders of the same type play an identical strategy to facilitate a comparison with a second-price auction.

Since the optimal bidding strategy of a bidder with value zero is straightforward, we consider a type-n bidder with value v_n. Let $F_n(x) \equiv \Pr(b \le x)$ denote the cumulative distribution function for a type-n bidder's bid when he has a positive value and let its support be $[\underline{b}_n, \overline{b}_n]$. We denote his expected payoff by V_n^F. The following observation provides the necessary conditions for an efficient equilibrium.

Lemma 1. *In any symmetric equilibrium,* $V_n^F > 0$. *Moreover, if the allocation is efficient in equilibrium, then the following results hold:*

(i) $\underline{b}_1 = 0$ *and* $\overline{b}_n = \underline{b}_{n+1}$.

(ii) $\underline{b}_n < \overline{b}_n$ *and* $F_n(\cdot)$ *is continuous and strictly increasing.*

Proof. See the Appendix. ■

The first part of Lemma 1 implies that $\overline{b}_n < v_n$ for any n in any symmetric

[5] See Section 4.3 for an analysis of inefficient equilibria in that allocations are not efficient.

equilibrium because any bidder with a positive value obtains a positive payoff. The second part of the lemma shows that in any (symmetric) efficient equilibrium, there cannot be a gap in the set of bids, and a bidder with higher value bids more aggressively. It also implies that no pure-strategy equilibrium exists. Another important implication of Lemma 1 is that for an efficient equilibrium to exist, there must be at least two bidders of each type, except for the lowest type; that is, $m_n \geq 2$ for any $n \geq 2$, as we have already assumed. To see this, suppose that $m_n = 1$ for some $n \geq 2$. Then, we must have $\underline{b}_n = \overline{b}_n$.[6] However, in this case, a type-$n-1$ bidder has an incentive to outbid the type-n bidder because $\overline{b}_{n-1} = \underline{b}_n$ and the deviation to slightly above $\underline{b}_n$ makes his winning probability jump, which is a contradiction.

The unique efficient equilibrium can be constructed from the following relationship between $\overline{b}_{n-1}$ and $\overline{b}_n$. A type-n bidder who bids $\overline{b}_{n-1} (= \underline{b}_n)$ wins if and only if no type weakly above n draws a positive value, and thus, his expected payoff is $Q_n r_n (v_n - \overline{b}_{n-1})$. Similarly, a type-$n$ bidder who bids $\overline{b}_n$ wins if and only if no type strictly above n draws a positive value, and thus, his expected payoff is $Q_n (v_n - \overline{b}_n)$. Because a type-$n$ bidder must be indifferent between $\overline{b}_{n-1}$ and $\overline{b}_n$, we have $Q_n r_n (v_n - \overline{b}_{n-1}) = Q_n (v_n - \overline{b}_n)$, implying that

$$\overline{b}_n = (1 - r_n) v_n + r_n \overline{b}_{n-1}.$$

Solving the difference equation with the initial value $\overline{b}_0 = 0$, we have

$$\overline{b}_n = (1 - r_n) v_n + r_n (1 - r_{n-1}) v_{n-1} + \cdots + \prod_{j=2}^{n} r_j (1 - r_1) v_1 + \prod_{j=1}^{n} r_j v_0. \tag{1}$$

Given the cutoffs $\{\overline{b}_1, \ldots, \overline{b}_n\}$, the expected payoff of a type-n bidder, $V_n^F = Q_n r_n (v_n - \overline{b}_{n-1})$, can be calculated immediately.

Next, to obtain a type-n bidder's bidding strategy, suppose that he bids some $b \in [\underline{b}_n, \overline{b}_n]$. Then, he outbids a rival of the same type when the latter draws zero (which happens with probability $1 - p_n$) or draws a positive value but bids less than b (which happens with probability $p_n F_n(b)$). Thus, the type-n bidder's expected payoff is equal to

$$Q_n (1 - p_n + p_n F_n(b))^{m_n - 1} (v_n - b),$$

where $F_n(b)$ is uniquely determined by equating this with V_n^F above.

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[6] If $m_n = 1$, then the winning probability of the type-n bidder does not change between $\underline{b}_n$ and $\overline{b}_n$. Hence, he can be indifferent between $\underline{b}_n$ and $\overline{b}_n$ only when $\underline{b}_n = \overline{b}_n$.

Proposition 2. *The unique symmetric efficient equilibrium is that each type-* n *bidder with value* v_n *bids according to the distribution function* $F_n(\cdot)$ *that satisfies*

$$(1-p_n+p_nF_n(b))^{m_n-1}(v_n-b)=(1-p_n)^{m_n-1}(v_n-\overline{b}_{n-1}).$$

A type- n *bidder's expected payoff in equilibrium is* $V_n^F=Q_nr_n(v_n-\overline{b}_{n-1})$.

We now characterize the condition under which the strategy profile described in Proposition 2 constitutes an equilibrium. To this end, we first consider one-step deviations: a one-step downward deviation of a type-$n+1$ bidder with value v_{n+1} to bid $b\in[\overline{b}_{n-1},\overline{b}_n)$ and a one-step upward deviation of a type-$n-1$ bidder with value v_{n-1} to bid $b\in(\overline{b}_{n-1},\overline{b}_n]$.[7] Later, we show that it suffices to consider such one-step deviations to verify whether any deviation is profitable.

Suppose that a type-$n+1$ bidder with value v_{n+1} deviates to bid $b\in(\overline{b}_{n-1},\overline{b}_n]$. He wins if and only if no other bidder of type strictly above n draws a positive value, and each type-n bidder bids less than b. Hence, his payoff from such a deviation is $Q_{n+1}r_{n+1}(1-p_n+p_nF_n(b))^{m_n}(v_{n+1}-b)$. For the deviation to not be profitable, this payoff must be lower than his equilibrium payoff, that is,

$$V_{n+1}^F=Q_{n+1}r_{n+1}(v_{n+1}-\overline{b}_n)\ge Q_{n+1}r_{n+1}(1-p_n+p_nF_n(b))^{m_n}(v_{n+1}-b). \tag{2}$$

From the fact that a type-n bidder is indifferent between $\overline{b}_n$ and $b\in[\overline{b}_{n-1},\overline{b}_n)$, we also have

$$Q_n(v_n-\overline{b}_n)=Q_n(1-p_n+p_nF_n(b))^{m_n-1}(v_n-b).$$

Using this, condition (2) is written equivalently as

$$\frac{v_{n+1}-\overline{b}_n}{v_{n+1}-b}\ge(1-p_n+p_nF_n(b))\frac{v_n-\overline{b}_n}{v_n-b}.$$

Note that since $1-p_n+p_nF_n(b)<1$ and $\frac{v_{n+1}-\overline{b}_n}{v_{n+1}-b}>\frac{v_n-\overline{b}_n}{v_n-b}$ for any $b\in[\overline{b}_{n-1},\overline{b}_n)$, the inequality always strictly holds, which shows that no bidder has an incentive to deviate one step below.

Next, consider a type-$n-1$ bidder's deviation to bid $b\in(\overline{b}_{n-1},\overline{b}_n]$. His payoff from such a deviation is $Q_n(1-p_n+p_nF_n(b))^{m_n}(v_{n-1}-b)$, because the bidder wins if and only if no bidder of type strictly above n draws a positive value and each type-

[7] It is clear that bidders with zero value have no incentive to deviate.

n bidder bids less than b. For the bidder not to deviate, we must have

$$V_{n-1}^{F}=Q_{n-1}(v_{n-1}-\overline{b}_{n-1})\geq Q_{n}(1-p_{n}+p_{n}F_{n}(b))^{m_{n}}(v_{n-1}-b).$$

We can rewrite this as

$$\left(\frac{v_{n-1}-\overline{b}_{n-1}}{v_{n-1}-b}\right)^{m_n-1}\geq\left(\frac{v_{n}-\overline{b}_{n-1}}{v_{n}-b}\right)^{m_n}, \tag{3}$$

using the relation $Q_{n}(1-p_{n})^{m_{n}-1}(v_{n}-\overline{b}_{n-1})=Q_{n}(1-p_{n}+p_{n}F_{n}(b))^{m_{n}-1}(v_{n}-b)$, which is obtained from the fact that a type-n bidder is indifferent between $\overline{b}_{n-1}$ and $b\in(\overline{b}_{n-1},\overline{b}_{n}]$. In Proposition 3, we show that condition (3) is equivalent to

$$v_{n-1}\leq\frac{(m_{n}-1)v_{n}+\overline{b}_{n-1}}{m_{n}}, \tag{4}$$

where $\overline{b}_{n-1}$ is given by (1).

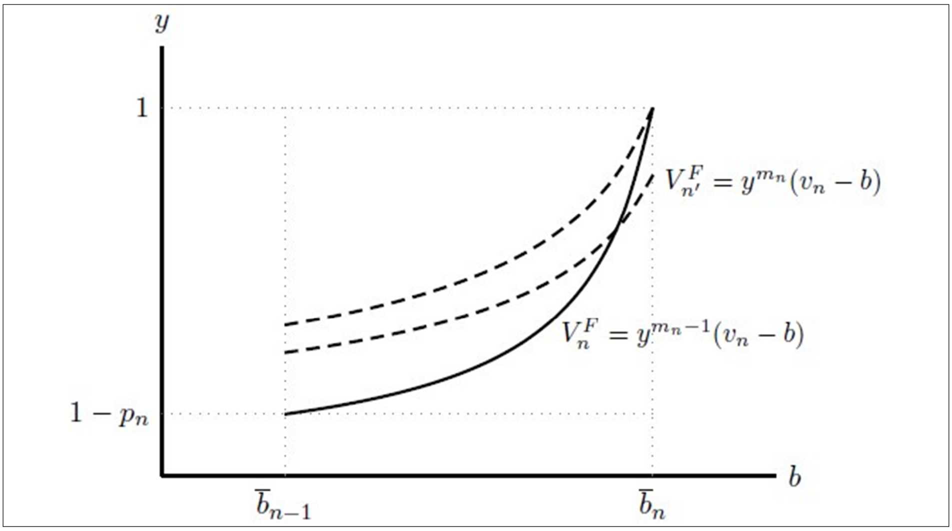
Note that since (2) is always satisfied, no bidder has an incentive for one-step deviations if and only if (4) holds. However, this only ensures "local" incentive compatibility, and there is a concern that a type-n bidder may profitably deviate "globally" by bidding higher than $\overline{b}_{n+1}$ or lower than $\underline{b}_{n-1}$. Proposition 3 ensures that no such global deviation is profitable as long as (4) is satisfied.

Proposition 3. *A symmetric efficient equilibrium exists in the first-price auction if and only if (4) holds for any* $n=2,\dots,N$.

Proof. See the Appendix. ■

Why do bidders have no incentive for downward deviation but may have an incentive for upward deviation? To address this question, consider two bidders, one of type n and the other of type n', and suppose that both bidders bid the same price $b\in[\overline{b}_{n-1},\overline{b}_{n}]$. The two bidders have two differences. First, they have different values of v_n and $v_{n'}$. This is familiar in symmetric auctions, and the bidder with the higher value tends to bid the higher price. Second, they also have different beliefs on the types of competitors. The type-n bidder knows that at least one type-n bidder (i.e., himself) has a positive value, whereas the type-n' bidder does not. Thus, the probability of outbidding other type-n bidders is $(1-p_{n}+p_{n}F_{n}(b))^{m_{n}-1}$ to the former, but is $(1-p_{n}+p_{n}F_{n}(b))^{m_{n}}$ to the latter. This difference results in the latter bidder having a stronger incentive to bid aggressively in the range $[\overline{b}_{n-1},\overline{b}_{n}]$.

[Figure 1] Indifference curves of a type-n bidder (solid) and a type-n' bidder (dashed) when, hypothetically, they have the same value.



To see this point, suppose hypothetically, that the type-n' bidder has the same value as the type-n bidder (i.e., $v_{n'} = v_n$), and denote by $y \equiv 1 - p_n + p_n F_n(b)$ the probability that a type-n bidder bids less than or equal to b. Then, the expected payoffs of type-n and type-n' bidders are $V_n^F = y^{m_n-1}(v_n - b)$ and $V_{n'}^F = y^{m_n}(v_n - b)$, respectively. Due to the difference-in-beliefs, the type-n bidder's marginal rate of substitution between b and y is

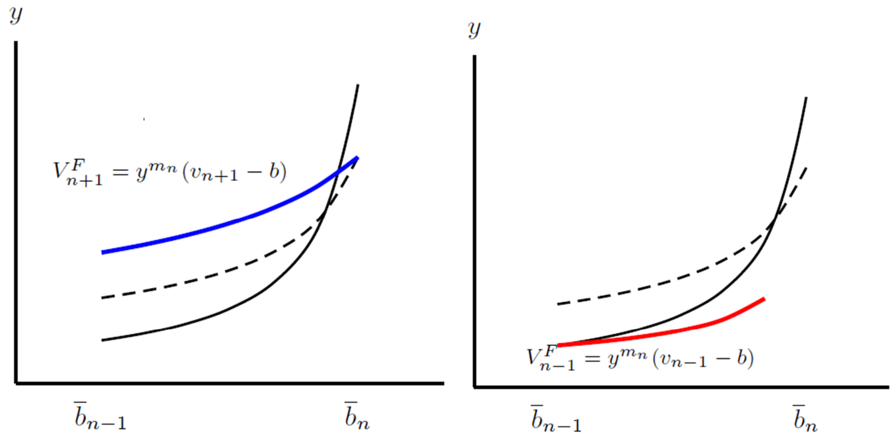
$$MRS_{b,y}^{n} = -\frac{y}{(m_n - 1)(v_n - b)},$$

whereas the type-n' bidder's is

$$MRS_{b,y}^{n'} = -\frac{y}{m_n(v_n - b)}.$$

Figure 1 depicts the indifferent curves of the two bidders. Observe that the type-n' bidder's indifference curve is flatter than that of the type-n bidder. Therefore, if the type-n bidder is indifferent over the interval $[\bar{b}_{n-1}, \bar{b}_n]$, then the optimal price for the type-n' bidder is the highest price $\bar{b}_n$. Intuitively, the type-n bidder has better information than the other types for the bids in $(\bar{b}_{n-1}, \bar{b}_n]$. Since he can better tailor his bid, he tends to bid just enough to win the auction, whereas the type-n' bidder cannot fine-tune his bid and, therefore, tends to bid higher.

[Figure 2] Indifference curves on $[\overline{b}_{n-1}, \overline{b}_n]$



(a) Type-$n+1$ bidder (b) Type-$n-1$ bidder

Note that difference-in-values and the difference-in-beliefs work in the same direction for downward deviations. To illustrate, consider a type-$n+1$ bidder. The difference-in-values provides him with an incentive to outbid type-n bidders even if they have the same beliefs. The difference-in-beliefs strengthens this incentive further. Therefore, the type-$n+1$ bidder never has an incentive to deviate downward. Put differently, for $b \in [\overline{b}_{n-1}, \overline{b}_n]$, the marginal rate of substitution between y and b for the type-$n+1$ bidder is

$$MRS_{b,y}^{n+1} = -\frac{y}{m_n(v_{n+1} - 1)} > MRS_{b,y}^{n'}.$$

His indifference curve is even flatter than the type-n' bidder's, as depicted in Figure 2(a). Therefore, it follows that the optimal price in the interval $[\overline{b}_{n-1}, \overline{b}_n]$ is $\overline{b}_n$.

For upward deviations, the two differences work in the opposite direction, and this is why the incentive compatibility condition (4) may be violated. Consider a type-$n-1$ bidder. The difference-in-values weakens his incentive to deviate upward, whereas the difference-in-beliefs encourages the upward deviation. Depending on the parameter values, the latter force may outweigh the former, in which case we do not have an efficient equilibrium. More precisely, the marginal rate of substitution between y and b for the type-$n-1$ bidder is

$$MRS_{b,y}^{n-1} = -\frac{y}{m_n(v_{n-1} - b)},$$

which may or may not be smaller than $MRS^{n}_{b,y}$. In particular, the type-$n-1$ bidder has an incentive to deviate to $[\overline{b}_{n-1}, \overline{b}_{n}]$ if and only if

$$MRS^{n-1}_{b,y} > MRS^{n}_{b,y} \text{ when } b = \overline{b}_{n-1} \Leftrightarrow v_{n-1} > \frac{(m_n - 1)v_n + \overline{b}_{n-1}}{m_n},$$

as depicted in Figure 2(b), which explains condition (4).

Observe that condition (4) holds when v_n is sufficiently larger than v_{n-1} or when m_n is sufficiently large. This is intuitive. When v_n is sufficiently larger than v_{n-1}, the difference-in-value effect dominates and when m_n is sufficiently large, the difference-in-belief effect is small because the difference between $y^{m_n - 1}$ and y^{m_n} decreases as m_n increases. See Section 4.1. for details.

3.3. Revenue Comparison

This section compares the revenues of the first- and second-price auctions in the class of equilibria with efficient allocation.

Proposition 4. *The seller's expected revenue is higher in the second-price auction than it is in the first-price auction with an efficient equilibrium.*

Proof. It is clear that $V^F_1 = V^S_1 = Q_1 r_1 v_1$. That is, the lowest type receives the same expected payoff in the two auction formats. We show that any other type obtains a strictly higher expected payoff in the first-price than in the second-price auction, thereby proving that the seller's revenue is higher in the latter. To this end, fix $n > 1$. Then, from Proposition 1,

$$V^S_n = Q_n r_n (v_n - (1 - q_{n-1}) v_{n-1} - q_{n-1}(1 - q_{n-2}) v_{n-2} - \cdots - q_{n-1} \ldots q_1 v_0).$$

Now, from Proposition 2, we have

$$V^F_n = Q_n r_n (v_n - (1 - r_{n-1}) v_{n-1} - r_{n-1}(1 - r_{n-2}) v_{n-2} - \cdots - r_{n-1} \ldots r_1 v_0).$$

It suffices to show that

$$\begin{aligned} &(1 - q_{n-1}) v_{n-1} + q_{n-1}(1 - q_{n-2}) v_{n-2} + \cdots + q_{n-1} \ldots q_1 v_0 \\ &> (1 - r_{n-1}) v_{n-1} + r_{n-1}(1 - r_{n-2}) v_{n-2} + \cdots + r_{n-1} \ldots r_1 v_0 \quad . \end{aligned} \tag{5}$$

Observe that both $(q_{n-1} \ldots q_1, \cdots, q_{n-1}(1 - q_{n-2}), 1 - q_{n-1})$ and $(r_{n-1} \ldots r_1, \cdots, r_{n-1}(1 -$

$r_{n-2}),1-r_{n-1})$ are well-defined probability vectors.[8] Because $r_n = q_n/(1-p_n) > q_n$ for any n,

$$1-q_{n-1}\ldots q_k > 1-r_{n-1}\ldots r_k \quad \text{for any} \quad k=1,\ldots,n-1.$$

Therefore, the former probability vector first-order stochastically dominates the latter and the result follows immediately. ■

In the absence of ex-ante asymmetries, the bidders' beliefs differ only in terms of their realized values. However, with ex-ante asymmetries among the bidders, their beliefs differ in one other respect, as explained previously, which makes bidders extract additional information rents in the first-price auction but not in the second-price auction. Thus, it follows that without distributive distortions in an efficient equilibrium, bidders enjoy a higher expected payoff in the first-price auction than they do in the second-price auction and accordingly, the seller's expected revenue is lower in the former than in the latter.

IV. Discussion

4.1. A Large Number of Bidders

The difference-in-beliefs is important to obtaining the revenue ranking in Proposition 4, and thus, a natural question is under what circumstances the difference-in-beliefs disappears and, if so, what the revenue ranking would be. In this section, we show that as the number of bidders for each type increases, the difference-in-beliefs vanishes and the revenue equivalence is restored.

Recall that the difference-in-beliefs means a type-n bidder with a positive value knows that at least one of the type-n bidders (i.e., himself) has a positive value, whereas other types of bidders do not know. Although this additional information plays an important role when m_n is small, its effect will reduce as m_n grows. Thus, it is intuitive that the two auction formats yield the same revenue in the limit as m_n goes to infinity.

To be precise, let us increase m_n but, at the same time, proportionally decrease p_n so that the expected number of type-n bidders with positive values, $m_n p_n$, stays constant. Note that the level of competition among bidders does not change as m_n increases, whereas both $y^{m_n-1} = (1-p_n + p_n F_n(b))^{m_n-1}$ and $y^{m_n} = (1-p_n + p_n F_n(b))^{m_n}$ converge to $e^{-m_n p_n (1-F_n(b))}$, and thus, $y^{m_n-1} - y^{m_n}$ decreases. That is, the

[8] This is because $1-x_{n-1}+x_{n-1}(1-x_{n-2})+\cdots+x_{n-1}\ldots x_1 = 1$ for $0 \le x_i \le 1$ and $i=1,\ldots,n$.

difference-in-beliefs vanishes as m_n goes to infinity. Condition (4) never fails in the limit, and so the efficient equilibrium always exists in the first-price auction. Note also that as m_n tends to infinity (while keeping $m_n p_n$ constant), both $r_n = (1-p_n)^{m_n - 1}$ and $q_n = (1-p_n)^{m_n}$ converge to $e^{-m_n p_n}$ and, therefore, inequality in (5) becomes an equality in the limit. This implies that all bidders' expected payoffs become identical in the two auction formats and, hence, the revenue equivalence follows.

4.2. Continuous Distributions

Our model assumes binary distribution functions for each type of bidders' valuation. In this section, we study how the results in our setting are related to the findings in a continuous model in the asymmetric auction literature. The closest related works are those of Hubbard and Kirkegaard (2015) and Kirkegaard (2016), who consider an environment with two bidder types and multiple bidders for each type.

Let $G_n(\cdot)$ be a cumulative distribution function of valuation with support $[\underline{v}_n, \overline{v}_n]$, where $n = 1,2$ and $0 \le \underline{v}_1 \le \underline{v}_2 < \overline{v}_1 < \overline{v}_2$. There are m_n bidders who draw their values from G_n. Hubbard and Kirkegaard (2015) show that in a symmetric equilibrium with a bidding function $b_n(\cdot)$, any bid b made by a type-n bidder satisfies $b \in [\underline{b}, \overline{b}_n]$ for some $\underline{b}$ and $\overline{b}_n \equiv b_n(\overline{v}_n)$. Defining $\hat{v}$ such that $b_2(\hat{v}) = \overline{b}_1$, they show that $\hat{v} < \overline{v}_2$ if and only if

$$m_2 \overline{v}_1 - (m_2 - 1)\overline{v}_2 < \overline{b}_1. \tag{6}$$

If condition (6) is satisfied, then type-2 bidders with a value above $\hat{v}$ do not face a type-1 rival. Thus, there is a range of bids on which only type-2 bidders submit bids, called "bid-separation."[9] Kirkegaard (2016) extends these results and compares a range of reserve prices for which the first-price auction yields a higher revenue than that of the second-price auction.

Although the assumptions in Hubbard and Kirkegaard (2015) and Kirkegaard (2016) do not allow for a discrete model, we may consider a continuous approximation of our binary distributions. Let $\overline{v}_n = v_n$ and $\underline{v}_n = 0$ for all $n = 1,2$, and let G_n be any continuous distribution that approximates well our discrete setting, so that $G_n(v_n - \epsilon) = 1 - p_n$ for some $\epsilon > 0$. It immediately follows that condition (6) is the same as (4), a necessary and sufficient condition for an efficient equilibrium in our discrete model. As bidders with a value above $\hat{v}$ compete

[9] Note that type-2 bidders with a value below $\hat{v}$ engage in competition with type-1 bidders and may lose against them. If condition (6) fails, then $\hat{v} = \overline{v}_2$ and so $\overline{b}_2 = \overline{b}_1$. Thus, there is no bid-separation in this case.

against rivals with a value above $\hat{v}$ when (6) holds in Hubbard and Kirkegaard (2015), bidders with a value v_n compete against rivals with the same value when (4) holds in our model. Thus, an efficient allocation in our model can be viewed as an extreme form of bid-separation. Next, Kirkegaard (2016) assumes a hazard-rate dominance of distributions (i.e., $G_2(v)/G_1(v)$ is non-decreasing) to compare the revenues of the two auction formats. However, our approximation of distributions does not satisfy this assumption,[10] and so the results of revenue comparisons in Kirkegaard (2016) do not follow immediately.[11]

Nevertheless, it is worth noting that the number of bidders affects the revenue ranking both in the model of Kirkegaard (2016) and in our model. Hubbard and Kirkegaard (2015) and Kirkegaard (2016) show that $\hat{v} \to v_1$ as $m_1 \to \infty$ or $m_2 \to \infty$, implying that bid-separation is more likely to occur as the number of bidders increases. Note that as the competition intensifies, bidders' bids become close to their values. From the type-2 bidders' perspective, this means that v_1 plays a role similar to that of a reserve price because a bid below v_1 almost certainly results in a loss. Thus, the competition will effectively take place among type-2 bidders and exclude type-1 bidders. Recall that in our model, the efficient allocation does not induce revenue equivalence, per se. However, as m_n increases, the difference-in-beliefs vanishes and the revenue equivalence is restored in the limit, as explained in Section 4.1. In summary, increasing the number of bidders makes competing bidders effectively "symmetric" for both models, although the exact mechanisms differ.

4.3. Inefficient Equilibria

In this section, we consider equilibria with inefficient allocations in first-price auctions and compare the revenues of the two auction formats. Because it appears difficult to fully characterize an equilibrium for arbitrary numbers of bidders with different types, we consider the case that $N = 2$ and $m_n = 2$ for all $n = 1,2$ and show that the revenue ranking can go either way.

Consider the first-price auction. Let F_n be the cumulative distribution function for a type-n bidder's bid in equilibrium when he has a positive value, and V_n^F denote his payoff; that is

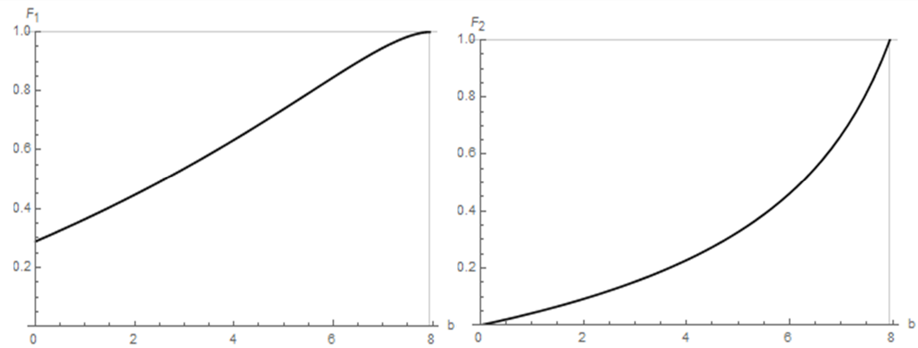
$$V_n^F = (1 - p_n + p_n F_n(b))(1 - p_{n'} + p_{n'} F_{n'}(b))^2 (v_n - b),$$

[10] Note that $G_2(v)/G_1(v) = (1-p_2)/(1-p_1)$ for $v \in [0, v_1]$; $G_2(v)/G_1(v) = 1 - p_2$ for $v \in [v_1, v_2)$; and $G_2(v)/G_1(v) = 1$ for $v \geq v_2$. Thus $G_2(v)/G_1(v)$ can be decreasing (because $(1-p_2)/(1-p_1) > 1 - p_2$). See Doni and Menicucci (2013) for a similar comparison of their model with that of Kirkegaard (2012).

[11] Kirkegaard (2016) assumes that the seller uses a reserve price and also compares the optimal reserve prices for the two auction formats. However, we do not consider a reserve price.

for any bid b, where $n'=1,2$ and $n'\neq n$. In what follows, we look for an equilibrium in which both F_1 and F_2 have common support $[0,\overline{b}]$ for some $\overline{b}$. This implies that there is a positive chance that a bidder with value v_1 wins against a rival with v_2, although $v_1<v_2$. That is, the allocation is not efficient.

[Figure 3] Equilibrium bid distributions for bidder types in Example 1



(a) $F_1(b)$ (b) $F_2(b)$

Example 1. *Let* $v_1=10$, $v_2=12$, $p_1=0.25$, *and* $p_2=0.5$. *Then, there exists an equilibrium with* $\overline{b}=7.94$ *in the first-price auction.*

Equilibrium bid distributions F_1 and F_2 in Example 1 are depicted in Figure 3, where $F_1(0)=0.2888$ and $F_2(0)=0$. We denote the social surplus and the seller's revenue in the first-price auction by S^F and R^F, respectively. It can be calculated that $S^F=9.974$, $V_1^F=2.055$, and $V_2^F=4.055$. Because the seller's revenue is the difference between the social surplus and bidders' payoff and bidders with zero value earn a zero payoff, we have $R^F=S^F-2(p_1V_1^F+p_2V_2^F)=4.891$. Next, for the second-price auction, we can show that the social surplus is $S^S=10.093$, and each type of bidder's payoff when he has a positive value is $V_1^S=1.875$ and $V_2^S=3.812$, using the fact that it is a weakly dominant strategy for each bidder to bid his value. Thus, the seller's revenue is $R^S=S^S-2(p_1V_1^S+p_2V_2^S)=5.344$. Note that compared with the second-price auction, the first-price auction entails a lower social surplus, $S^F<S^S$, because of the inefficient allocation, whereas the bidders' payoff is higher for both types, $V_n^F>V_n^S$ for all $n=1,2$. Thus, it follows that the seller's revenue is higher in the second-price auction than in the first-price auction, $R^F<R^S$.

Example 2. *Let* $v_1=10$, $v_2=11.5$, $p_1=0.3$, *and* $p_2=0.6$. *Then, there exists an equilibrium with* $\overline{b}=8.851$ *in the first-price auction.*[12]

[12] Equilibrium bid distributions F_1 and F_2 are drawn similarly to Figure 3, with $F_1(0)=0.059$

The first-price auction in Example 2 yields that $S^F = 10.077$, $V_1^F = 1.148$, and $V_2^F = 2.298$. Thus, the seller's revenue in the first-price auction is $R^F = 6.629$. For the second-price auction, we have $S^S = 10.182$, $V_1^S = 1.12$, $V_2^S = 2.42$ and so $R^S = 6.606$. Observe that although the inefficiency in allocation implies a lower social surplus in the first-price auction than in the second-price auction ($S^F < S^S$), the seller's revenue is higher in the former than in the latter ($R^F > R^S$), because the bidders enjoy a higher payoff in the second-price auction.

V. Conclusion

We have identified a source of inefficiency in first-price auctions. Our results are driven by the difference in the beliefs of different bidder types, which is a consequence of ex-ante bidder asymmetries. As noted previously, our results also help us to understand the existing results, including those of an inefficient equilibrium, continuous distributions, and the role of increased competition, in the asymmetric auction literature. We believe our results provide valid insights into the roles of bidder asymmetries in auctions.

$> 0 = F_2(0)$.

Appendix: Omitted Proofs

Proof of Lemma 1. For the first part of the lemma, observe that each bidder has a positive probability of being the only bidder with a positive value. Hence, a type-n bidder's expected payoff is at least as much as $Q_0 v_n/(1-p_n)>0$.

Consider any efficient equilibrium. Observe that since a higher type always bids more than a lower type, $\overline{b}_n \le \underline{b}_{n+1}$ for any $n=1,\dots,N-1$. Now, we show that $F_n(\cdot)$ is continuous and strictly increasing. Suppose to the contrary that there is an atom in the support of $F_n(\cdot)$ (which includes the case where $\underline{b}_n=\overline{b}_n$). If a type-$n$ bidder bids just above the point with an atom, then he obtains strictly more than his equilibrium payoff because it makes his winning probability jump, whereas his payment (which, by the first part of the lemma, is strictly smaller than v_n) increases only marginally. This proves that, in equilibrium, $F_n(\cdot)$ cannot have any atom, which also implies that $\underline{b}_n<\overline{b}_n$ and is a continuous function. Next, suppose that $F_n(\cdot)$ is constant over some interval, say $[b',b'']\subset[\underline{b}_n,\overline{b}_n]$. Then, a bidder of type-n strictly prefers bidding b' to b'', because it does not change his winning probability but lowers his payment.[13] Therefore, $F_n(\cdot)$ must be strictly increasing in its support.

Lastly, we show that $\overline{b}_n=\underline{b}_{n+1}$. To see this, suppose $\overline{b}_n<\underline{b}_{n+1}$. Given that $F_{n+1}(\cdot)$ has no atom, a type-$n+1$ bidder who bids $\underline{b}_{n+1}$ wins only when no bidder of the same type draws a positive value. Therefore, his bid must be optimal conditional on this event. Because his winning probability does not change between $\overline{b}_n$ and $\underline{b}_{n+1}$, his bid must be arbitrary close to $\overline{b}_n$. The same reasoning applies to $\underline{b}_1=0$. ■

Proof of Proposition 3. We first show that (3) is equivalent to (4). Because (3) holds with equality when $b=\overline{b}_{n-1}$, a necessary condition is that the derivative of LHS of (3) at $\overline{b}_{n-1}$ is no less than that of the RHS; that is,

$$\left(\frac{v_{n-1}-\overline{b}_{n-1}}{v_{n-1}-\overline{b}_{n-1}}\right)^{m_n-1}\frac{m_n-1}{v_{n-1}-\overline{b}_{n-1}}=\frac{m_n-1}{v_{n-1}-\overline{b}_{n-1}}\ge\left(\frac{v_n-\overline{b}_{n-1}}{v_n-\overline{b}_{n-1}}\right)^{m_n}\frac{m_n}{v_n-\overline{b}_{n-1}}=\frac{m_n}{v_n-\overline{b}_{n-1}}.$$

Rearranging the terms, we have

$$v_{n-1}\le\frac{(m_n-1)v_n+\overline{b}_{n-1}}{m_n}.$$

[13] Because we focus on a symmetric equilibrium, no bidder of the same type bids in $[b',b'']$. Moreover, because $\underline{b}_n\le\overline{b}_{n+1}$ for all n, no bidders of different types bid in $[b',b'']$, as well.

This condition is also sufficient because

$$\left(\frac{v_{n-1}-\overline{b}_{n-1}}{v_{n-1}-b}\right)^{m_n-1}\frac{m_n-1}{v_{n-1}-b}\geq\left(\frac{v_n-\overline{b}_{n-1}}{v_n-b}\right)^{m_n}\frac{m_n}{v_n-b}\quad\text{for } b>\overline{b}_{n-1},$$

whenever $\left(\frac{v_{n-1}-\overline{b}_{n-1}}{v_{n-1}-b}\right)^{m_n-1}=\left(\frac{v_n-\overline{b}_{n-1}}{v_n-b}\right)^{m_n}$ and (4) holds. This means that the LHS of (3) crosses the RHS from below. Since (3) holds with equality at $b=\underline{b}_{n-1}$, the result follows.

We now show that any global deviation is not profitable whenever a bidder finds local deviations are not profitable. Suppose a type-$n+1$ bidder has a strict disincentive to deviate to $b\in[\overline{b}_{n'},\overline{b}_n)$ for some $n'<n-1$ and consider his deviation to $b\in[\overline{b}_{n'-1},\overline{b}_{n'}]$. His expected payoff from the deviation is equal to

$$\frac{Q_{n'}}{1-p_{n+1}}(1-p_{n'}+p_{n'}F_{n'}(b))^{m_{n'}}(v_{n+1}-b).$$

Since the bidder does not have an incentive to deviate to $\overline{b}_{n'}$, it suffices to show that this payoff is less than the bidder's deviation payoff to $\overline{b}_{n'}$; that is,

$$\frac{Q_{n'}}{1-p_{n+1}}(v_{n+1}-\overline{b}_{n'})>\frac{Q_{n'}}{1-p_{n+1}}(1-p_{n'}+p_{n'}F_{n'}(b))^{m_{n'}}(v_{n+1}-b).$$

Rearranging the terms as in the local deviation case,

$$\frac{v_{n+1}-\overline{b}_{n'}}{v_{n+1}-b}>(1-p_{n'}+p_{n'}F_{n'}(b))\frac{v_{n'}-\overline{b}_{n'}}{v_{n'}-b}.$$

By the same reasoning as in the local deviation case, this inequality always holds.

Next, suppose that (i) a type-n bidder has no incentive to deviate to $\overline{b}_{n+1}$ and (ii) a type-$n+1$ bidder has no incentive to deviate to $b>\overline{b}_{n+1}$. We prove that a type-n bidder also has no incentive to deviate to b. Since the result is obvious if $b\geq v_n$, suppose $b<v_n$. Let $R_k(b)$ be the probability that a type-k bidder wins when he bids b. Then, (ii) implies that

$$V^F_{n+1}=Q_{n+1}(v_{n+1}-\overline{b}_{n+1})\geq R_{n+1}(b)(v_{n+1}-b).$$

Since $b>\overline{b}_{n+1}$, $R_{n+1}(b)>Q_{n+1}$ and $R_n(b)=R_{n+1}(b)$. Together, they imply that

$$Q_{n+1}(v_n - \overline{b}_{n+1}) \geq R_n(b)(v_n - b).$$

By (i), a type-n bidder's expected payoff exceeds $Q_{n+1}(v_n - \overline{b}_{n+1})$. Therefore, he has no incentive to deviate to $b > \overline{b}_{n+1}$. ■

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