Functional Separability, Derived Demand, And The Elasticities of Substitution

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1. Introduction

Since Hicks (1932) and Robinson (1933) introduced the concept, elasticity of substitution has played a key role in the analysis of a production function. Hicks defined the elasticity of substitution in the case of two factors and constant returns to scale as $\delta = f_1 \cdot f_2 / f \cdot f_{12}$, where f_i and f_{ij} are the first and second partial derivatives of the production function. The elasticity of substitution gives a local description of an isoquant of a production functon, in terms independent of the units in which inputs are measured. In Hicks' phrase, it is "a measure of the ease with which the varying factors can be substituted for others"[1].

Robinson suggested that the elasticity of substitition is closely related to the cross price elasticity of factor demand in a competitive industry. She wrote: "When wages are reduced output will be increased. But the amount of labor employed per unit of output will also be increased. There are therefore two opposite influences on the aggregate amount of capital employed. Insofar as output increases there will be a tendency for the amount of capital to increase, but insofar as the amount of labor employed per unit of output increases, there will be a tendency for the amount of capital to be reduced. Now the increase in output will be greater the greater the elasticity of demand for the commodity, and the increase in the amount of labor per unit of output will be greater the greater the elasticity of substitution."[2] In this way the output effect, together with the substitution effect, determines the final combination of factors purchased when the

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1. See Hicks (1932), p. 117.
2. See Robinson (1933), p. 258.

price of one factor changes.

Allen (1938) generalized the Robinson's observation to the case of more than two factors, and expressed the relationship between a cross factor price elasticity and the elasticity of substitution with a mathematical equation, $E_{ij} = S_j(\sigma^A_{ij} - \eta)$. In the equation, E_{ij} is the cross factor price elasticity, S_j is the cost share of factor j, σ^A_{ij} is the Allen partial elasticity of substitution between factors i and j, and η is the own price elasticity of output demand.

Many years later, Sato (1967) contended that is the strongly separable production function in which the set of n inputs is partitoned into S subsets $[N_1, N_2, \ldots, N_s]$, the Allen partial elasticity of substitution is given by

$$\sigma^A_{ij} = \begin{cases} \sigma, & \text{if } i \in N_r,\ j \in N_s,\ r \neq s \\ \sigma + \frac{1}{\theta^s}(\sigma_s - \sigma), & \text{if } i, j \in N_s,\ i \neq j. \end{cases}$$

In the equation, θ^s is the relative expenditure share of the input group X^s, σ_s is intra-group elasticity of substitution, and σ is inter-group elasticity of substitution. Berndt and Christensen (1973) gave a proof of the eqality of inter-group Allen partial elasticities of substitution in a weakly separable production function, namely the first part of the Sato's equation.

In this paper I will derive Allen's and Sato's equations together following step-by-step the process of a firm's response to a factor price change. This process will yield a general formula for the case when the weak separability condition is not satisfied in the prduction function.

In section 2, Allen-Robinson formula for the elasticity of derived demand is proved based on Robinson's assertion. In section 3, Allen partial elasticities of substitution is reorganized into the combination of intro- and inter-group elasticities of substitution by tracing a firm's response to a factor price change and the result is compared with Sato's formula. In section 4, the components of the Allen partial elasticities of substitution is explored in detail when the separability condition is not assumed. Finally, concluding remarks follow in section 5.

2. Allen-Robinson Formula for the Elasticity of Derived Demand

2. 1. A Two-Factor Case

Suppose there is a competitive industry composed of n identical firms which are subject to the following assumptions:

Assumptions

1. The production function of a firm exhibits constant returns to scale.
2. Each firm has perfect information concerning the market equilibrium price and quantity.
3. The factor prices are independent of industry output level.
4. There is no entry or exit in the industry.

Under these assumptions, the following relation holds:

$$E_{KL} = S_L\ (\sigma_{KL} - \eta), \tag{1}$$

where E_{KL} is the output-variable cross price elasticity of demand for capital with respect to the price of labor, S_L is the cost share of labor, σ_{KL} is the elasticity of substitution between capital and labor, and η is the absolute value of the price elasticity of demand for output.

Proof

The assertion by Robinson can be represented by the following equation:

$$\frac{dK}{dW} = \left(\frac{\partial K}{\partial W}\right)_{y=y_o} + \frac{\partial K}{\partial y} \cdot \frac{\partial y}{\partial Y} \cdot \frac{\partial Y}{\partial P_Y} \cdot \frac{\partial C'}{\partial W}, \tag{2}$$

where W is wage rate, K is capital, C′ is unit total cost, P_Y is the competitive equilibrium price of output which is equal to the unit total cost C′, Y is industry output level, and y is firm output level. By multiplying both sides of the equation by W/K and rearranging, we get

$$\frac{dK}{dW} \cdot \frac{W}{K} = \left(\frac{\partial K}{\partial W} \cdot \frac{W}{K}\right)_{y=y_o} + \frac{\partial K}{\partial y} \cdot \frac{\partial y}{\partial Y} \cdot \frac{\partial Y}{\partial P_Y} \cdot \frac{\partial C'}{\partial W} \cdot \frac{W}{K} \cdot \frac{P_Y}{y} \cdot \frac{y}{P_Y}.$$

As $\frac{\partial C'}{\partial W} \cdot \frac{W}{P_Y} = S_L$ ($\frac{\partial C'}{\partial W} = \frac{L}{y}$ by Shepard's lemma),

$$\frac{\partial Y}{\partial P_Y} \cdot \frac{P_Y}{y} = -n \cdot \eta, \quad \text{and} \quad \frac{\partial K}{\partial y} \cdot \frac{y}{K} \cdot \frac{\partial y}{\partial Y} = \frac{1}{n}, \tag{3}$$

equation (3) is simplified into

$$E_{KL} = E_{KL}|_{y=y_o} - S_L \cdot \eta. \tag{4}$$

Furthermore,

$$E_{KL}|_{y=y_o} = \left(\frac{\partial K}{\partial W} \cdot \frac{W}{K}\right)_{y=y_o} = \left(\frac{W \cdot L}{C} \cdot \frac{C \cdot C_{KL}}{C_K \cdot C_L}\right)_{y=y_o} = S_L \cdot \sigma_{KL}, \tag{5}$$

where C is the total cost of y, $C_L = \partial C/\partial W = L$, $C_K = \partial C/\partial R = K$, $C_{KL} = \partial C/\partial R \cdot \partial W$ (R is rental rate for capital service) and σ_{KL} is the Allen-Uzawa elasticity of substitution when there are two factors of production. Combining (4) and (5), we get

$$E_{KL} = S_L \ (\sigma_{KL} - \eta) . \qquad \text{Q. E. D.}$$

2. 2. A General Case

Allen defined partial elasticities of substitution between factors X_i and X_j (against all other factors) as

$$\sigma^A_{ij} = \frac{\sum_{i=1}^{n} X_i f_i}{X_i X_j} \cdot \frac{F_{ij}}{F} \qquad i, j=1, 2, \ldots, n \tag{6}$$

where

$$F = \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

and F_{ij} is the cofactor of f_{ij} in F. When there are only two factors in the production function, the Allen partial elasticity of substitution becomes

$$\sigma^A_{12} = \frac{f_1 \ f_2 \ (X_1 \ f_1 + X_2 \ f_2)}{-X_1 X_2 \ (f_{11} \ f_2^2 - 2f_{12} \ f_1 \ f_2 + f_{22} \ f_1^2)} \tag{7}$$

which is the Robinson elasticity of substitution. When the production function exhibits constant returns to scale, $\sigma^A_{12} = f_1 \cdot f_2 / f \cdot f_{12}$, which is the Hicks elasticity of substitution. Even though Allen did not specifically indicate what should be held constant in defining the Allen partial elasticities of substitution, it is conventional to assume that output and the prices of factors other than X_j are to be held constant. A change in the price of one factor, holding output constant, is equivalent to a change in the relative price of the two factors in such a way that an increase in the price of one factor and a decrease in the price of the other rotate the isocost line along the same isoquant.

Allen derived the following aquation between the cross price elasticity of factor demand and the Allen partial elasticity of substitution:[3]

$$E_{ij} = S_j\,(\sigma_{ij}^{A} - \eta), \tag{8}$$

where σ_{ij}^{A} is the Allen paitial elasticity of substitution. Notice that Allen assumed constant returns to scale to derive the equation (8), while this assumption was not necessary to express the Allen partial elasticity of substitution by (6). As in the two-factor case, constant returns to scale is a necessary condition to derive the equation (8).

In the rest of this paper, we will investigate the relationship between the Allen partial elasticity of substitution and the cross price elasticity of factor demand by analyzing the firm behavior with two production functions characterized by different separability conditions: one which satisfies weak separability condition and the other which does not satisfy the condition. In the process we will examine the components of the Allen partial elasticities of substitution in more detail. As it was shown in the two-factor case that each firm acts as if it represents the industry under the given assumptions, the distinction between firm and industry will not be made in the following analysis.

3. Allen Partial Elasticities of Substitution under Weak Separability Assumption.

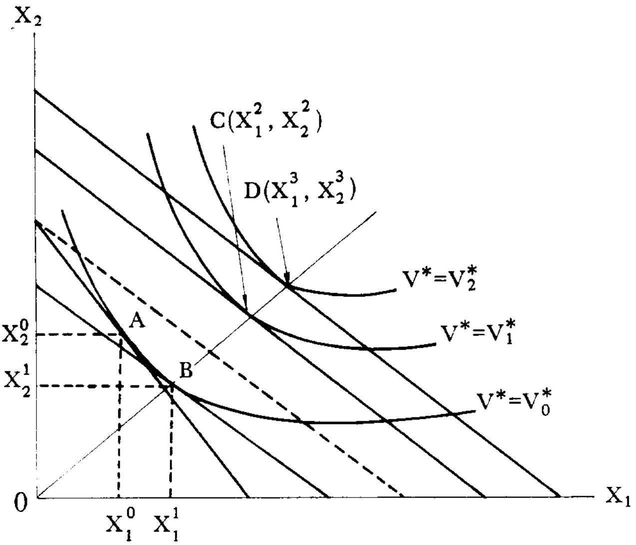
3. 1. Factor Price Change and a Firm's Response

Assume that in the production function $Y = F(X_1, X_2, X_3)$ which is twice-differentiable, strictly quasi-concave, and linear homogeneous in X's, X_1 and X_2 are weakly separable from X_3, so that the production function can be written as $Y = F'(V(X_1,X_2),X_3)$, where V satisfies the conditions above as well. Suppose the price of X_1 goes down while the prices of X_2 and X_3 remain the same. As shown in Figure 1, the optimal combination of two factors will change from A to B as long as V^* remains constant at V_0^*. But V^* does not remain constant because the decrease in the price of X_1 decreases the cost of the aggregate input V^*. If the cost of the aggregate input V^* decreases, the price line for output Y changes from $\bar{e}$ to F in Figure 2. Then the efficient factor combination of V^* and X_3 will change from E to F if the output level is held constant at $Y = Y_0$. In Figure 1, the increased use of V^* will expand the aggregate input isoquant to V_1^* and the factor combination will be X_1^2 and X_2^2. Even if we know that $X_1^2 > X_1^1 > X_1^0$ and $X_2^2 >$

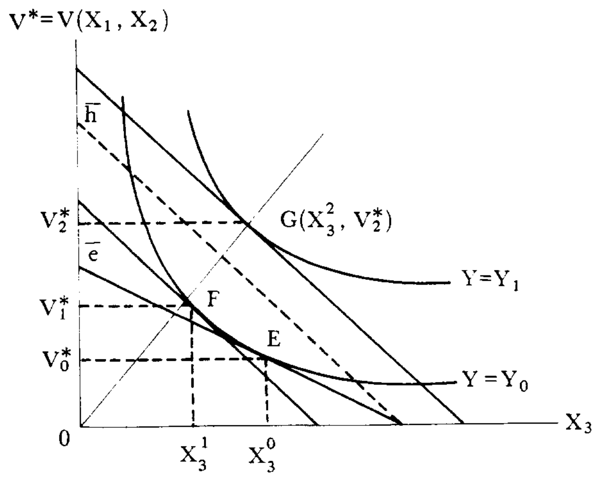
3. For a mathematical proof of this formula, refer Allen (1938), pp. 503-509.

X, we cannot predict *priori* which of $X_2^2$and X_2^0will be greater. It depends on the price elasticity of demand for V*.

In addition to the effects considered above, there is another effect caused by the fall in the price of X_1. If the cost of V* decreases, the price of Y will decrease in the competitive output market. At the lowered price of Y, more output will be sodl. To increase the output from Y_0 to Y_1 in Figre 2,



|Figure 1|



|Figure 2|

the utilization of V^* should increase from V_1^* to V_2^*. In Figure 1, to obtain the increased aggregate input V_2^*, the factor combination should change to X_1^3 and X_2^3. In short, the effect of the change in the price of X_1 on the derived demand for X_2 is the combination of the following three effects:

(1) the substitution effect holding V^* constant,
(2) the expansion effect with V^* variable, holding Y constant, and
(3) the output effect with Y variable.

Of the three effects, the substitution effect is positive; the expansion and output effects are both negative.

In the following analysis, we will investigate how the three effects described above are combined to measure the effect of a price change in X_1 on the utilization of X_2. The diagramatic analyses in Figures 2 and 3 can be expressed by the following equation:

$$\frac{dX_2}{dP_1} = \left(\frac{\partial X_2}{\partial P_1}\right)_{V^*=V_0^*} + \left(\frac{\partial X_2}{\partial V^*}\cdot\frac{\partial V^*}{\partial P_{V^*}}\cdot\frac{\partial C_{V^*}}{\partial P_1}\right)_{Y=Y_0} + \frac{\partial X_2}{\partial V^*}\cdot\frac{\partial V^*}{\partial Y}\cdot\frac{\partial Y}{\partial P_Y}\cdot\frac{\partial C_Y}{\partial P_{V^*}}\cdot\frac{\partial C_{V^*}}{\partial P_1}, \tag{9}$$

where C_{V^*} is the unit total cost of V^*, P_{V^*} is the implicit price of V^* which is equal to C_{V^*}. C_Y is the unit total cost of Y, and P_Y is the competitive equilibrium price of Y which is equal to C_Y. By multiplying both sides of the equation by P_1/X_2 and rearranging, we get

$$\frac{dX_2}{dP_1}\cdot\frac{P_1}{X_2} = \left(\frac{\partial X_2}{\partial P_1}\cdot\frac{P_1}{X_2}\right)_{V^*=V_0^*} + \left(\frac{\partial X_2}{\partial V^*}\cdot\frac{\partial V^*}{\partial P_{V^*}}\cdot\frac{\partial C_{V^*}}{\partial P_1}\cdot\frac{P_1}{X_2}\cdot\frac{P_{V^*}}{V^*}\cdot\frac{V^*}{P_{V^*}}\right)_{Y=Y_0} + \left(\frac{\partial X_2}{\partial V^*}\cdot\frac{\partial V^*}{\partial P_{V^*}}\cdot\frac{\partial C_{V^*}}{\partial P_1}\cdot\frac{P_1}{X_2}\right)\cdot\frac{\partial P_{V^*}}{V^*}\cdot\left(\frac{\partial V^*}{\partial Y}\cdot\frac{\partial Y}{\partial P_Y}\cdot\frac{\partial C_Y}{\partial P_{V^*}}\cdot\frac{P_{V^*}}{V^*}\cdot\frac{P_Y}{Y}\cdot\frac{Y}{P_Y}\right)\cdot\frac{V^*}{P_{V^*}}. \tag{10}$$

In equation (10), the following equalities hold:

$$\frac{\partial X_2}{\partial V^*}\cdot\frac{V^*}{X_2} = 1 \quad \text{from constant returns tro scale,}$$

$$\frac{\partial C_{V^*}}{\partial P_1}\cdot\frac{P_1}{P_{V^*}} = \frac{X_1}{V^*}\cdot\frac{P_1}{P_{V^*}} = S_{1V^*},$$

$$\frac{\partial V^*}{\partial P_{V^*}} \cdot \frac{P_{V^*}}{V^*} = E_{V^*V^*},$$

$$\frac{\partial C_Y}{\partial P_{V^*}} \cdot \frac{P_{V^*}}{P_Y} = \frac{V^*}{Y} \cdot \frac{P_{V^*}}{P_Y} = S_{V^*},$$

$$\frac{\partial V^*}{\partial Y} \cdot \frac{Y}{V^*} = 1 \text{ from constant returns to scale, and}$$

$$\frac{\partial Y}{\partial P_Y} \cdot \frac{P_Y}{Y} = E_{YY}.$$

Therefore, equation (10) is simplified into

$$E_{21} = E_{21}\big|_{V^*=V^*_o} + S_{1V^*} \cdot E_{V^*V^*}\big|_{Y=Y_o} + S_1 \cdot E_{YY}, \tag{11}$$

where S_{1V^*} is the cost share of X_1 in the total cost of aggregate input V^*, and S_1 is the cost share of X_1 in the total cost of Y. $E_{V^*V^*}$ and E_{YY} are the price elasticities of demand for V^* and Y, respectively.[4] Equation (11) shows that the output-variable cross price elasticity is the combination of three effects: the substitution effect, the expansion effect with output held constant, and the output effect induced by the change in output. Furthermore,

$$E_{21}\big|_{V^*=V^*_o} = \left(\frac{\partial X_2}{\partial P_1} \cdot \frac{P_1}{X_2}\right)_{V^*=V^*_o} = \left(C_{21} \cdot \frac{P_1}{C_2}\right)_{V^*=V^*_o}$$

$$= \left(\frac{P_1 \cdot C_1}{C} \cdot \frac{C \cdot C_{21}}{C_1 \cdot C_2}\right)_{V^*=V^*_o} = S_{1V^*} \cdot \sigma_{21}, \tag{12}$$

where C is the total cost of V^*, $C_i = \partial C / \partial P_i = X_i$, $C_{ij} = \partial^2 C / \partial P_i \cdot \partial P_j$, and σ_{21} is the elasticity of substitution between X_1 and X_2. So equation (11) can be written as

$$E_{21} = S_1 \left[\frac{1}{S_{V^*}}\left(\sigma_{21} + E_{V^*V^*}\big|_{Y=Y_o}\right) + E_{YY}\right], \tag{13}$$

where $S_{V^*} = S_1 / S_{1V^*}$. If the output is held constant,

$$E_{21}\big|_{Y=Y_o} = S_1 \left[\frac{1}{S_{V^*}}\left(\sigma_{21} + E_{V^*V^*}\big|_{Y=Y_o}\right)\right]. \tag{14}$$

Equation (14) shows that the cross price elasticity with constant output is the product of the cost share S_1 and the technical-condition term $1/S_{V^*}(\sigma_{21} + E_{V^*V^*}|_{Y=Y_o})$. The technical-condition term, which shows the

4. The process to get equation (11) is analogous to that of Berndt and Wood (1979).

degree of easiness with which two factors X_1 and X_2 substitute for each other while output Y is constant, is the Allen partial elasticity of substitution. In general,

$$\sigma_{ij}^{A} = \frac{1}{S_{V*}} (\sigma_{ij} + E_{V*V*}\big|_{Y=Y_0}), \qquad \text{if i and j come from some input group } V^*. \tag{15}$$

From equation (15) we see that the Allen partial elasticity of substitution is the sum of the intra-group elasticity of substitution and the expansion elasticity for V* divided by the cost share of V*.

3. 2. Comparison with Sato's Formula

Sato contended that if the production function is strongly separable, the Allen partial elasticity of substitution is given by

$$\sigma_{ij}^{A} = \sigma + \frac{1}{\theta^s} (\sigma_s - \sigma), \qquad \text{if i and j come from some input group } X^s. \tag{16}$$

In the equation, θ^s is the relative expenditure share of the input group X^s, and is the same as S_{V*} in equation (15); σ_s is the elasticity of sustitution within the input group X^s, and is the same as σ_{ij} in equation (15); and σ is the elasticity of substitution among input groups.[5] In the following, it will be proved that the Sato's equation is equivalent to equation (15) derived in this paper. Notice that the equation (15) does not require a two-level CES function which Sato employed. The weak separability condition is sufficient to get equation (15).

Proof

$$\sigma_{ij}^{A} = \frac{1}{\theta^s} (\sigma_s - \sigma) + \sigma = \frac{1}{S_{V*}} (\sigma_{V*} - \sigma) + \sigma = \frac{1}{S_{V*}} [\sigma_{V*} - (1 - S_{V*})\sigma].$$

As $S_{V*} + S_{R*} = 1$(by including all other inputs in the input group R*), the above equation can be written as

$$\sigma_{ij}^{A} = \frac{1}{S_{V*}} (\sigma_{V*} - S_{R*} \cdot \sigma).$$

Furthermore,

$$E_{V*V*} = \frac{\partial \ln V^*}{\partial \ln P_{V*}} = -(S_{R*} \cdot \sigma + S_{V*} \cdot \eta)^6.$$

If Y is constant $\eta = 0$, so $E_{V*V*}\big|_{Y=Y_0} = -S_{R*} \cdot \sigma.$

5. See Sato (1967), p. 203.
6. Refer Allen (1958), p. 373 for the proof of this relation.

Therefore,

$$\sigma_{ij}^{A} = \frac{1}{S_{V\bullet}}(\sigma_{V\bullet} - \sigma) + \sigma = \frac{1}{S_{V\bullet}}\left(\sigma_{V\bullet} + E_{V\bullet V\bullet}\big|_{Y=Y_o}\right). \qquad \text{Q.E.D.}$$

The elasticity of substitution with two factors, $\sigma_{V\bullet}$, is always positive, but the expansion elasticity is always negative, so the Allen partial elasticity of substitution, σ_{ij}^{A}, is always smaller than $\sigma_{V\bullet}$ when the factors i and j are weakly separable from other factors.

By combining equation (13) and (15), we get

$$E_{ij} = S_j(\sigma_{ij}^{A} + E_{YY}), \tag{17}$$

which is Allen's equation.

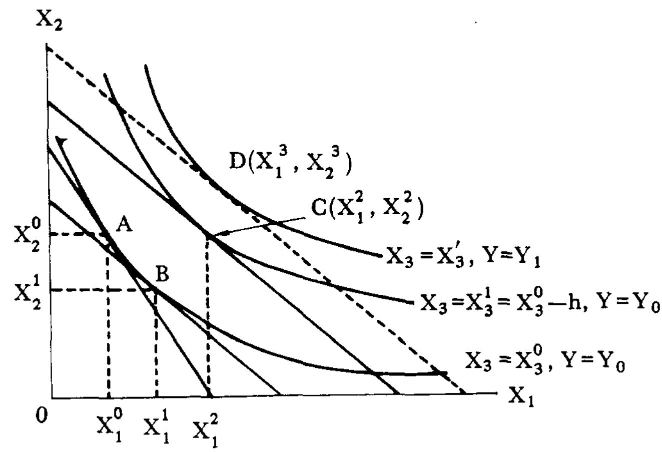
4. Generalization

Now we will investigate the case where the production function $Y=F(X_1,X_2,X_3)$, which is twice-differentiable, strictly quasi-concave, and linear homogeneous in X's, does not satisfy the separability condition. For the following diagramatic analysis, we will use iso-X_3 contours that keep the output level constant.[7] The farther the iso-X_3 curve moves from the origin, the lower level of X_3 is required to keep output constant. In Figure 3, $X_3=G(X_1,X_2;\ Y)$. At any given level of output Y, $X_3 = H(X_1, X_2)$.

If the price of X_1 goes down while the prices of X_2 and X_3 remain the same, the utilization of X_2 will be affected by the following three effects: the substitution effect, the third-factor effect, and the output effect. In Figure 3, the substitution effect is shown by the movement of the optimal point from A to B along the same level of X_3 and Y. This effect is caused by the increase in the relative price of X_2 to X_1. At the same time, the fall in the price of X_1 will increase the relative price of X_3 to X_1, changing the level of X_3 utilized. But the effect of the change in X_3 on the utilization of X_2 cannot be predicted a *priori* because the overall effect will be determined by simultaneous interactions among the three factors. Suppose the point C is chosen as the final combination of the three factors after all the interactions. Then X_1 and X_3 are substitutes because the lowered price of X_1 decreased the utilization of X_3. At point C, it is clear that $X_1^2 > X_1^1 > X_1^0$, but it is not known *a priori* whether X_2^2 or X_2^0 is greater. If X_2^2 is greater (smaller) than X_2^0, X_1 and X_2 are complements (substitutes).

In addition to the substitution and the third-factor effects discussed above, there is the output effect to consider. If the price of X_1 falls, the

7. The idea of this diagramatic analysis is adopted from Samuelson (1974), p. 1260.



[Figure 3]

production cost of output falls and the price of output falls in the competitive market. The increased production of output required for increased sales will increase the utilization of all three factors. This output effect on X_2 is shown by the movement of the optimal factor combination from the point C to the point D. Notice that the iso-quants Y_0 and Y_1 are located on different iso-Y planes. So the magnitude of X_3' cannot be compared with others in Figure 3.

The diagramatic explanation given above can be shown algebraically as follows:

$$\frac{dX_2}{dP_1} = \left(\frac{\partial X_2}{\partial P_1}\right)_{X_3 = X_3^0,\ Y = Y_0} + \left(\frac{\partial X_2}{\partial X_3} \cdot \frac{\partial X_3}{\partial P_1}\right)_{Y = Y_0} + \frac{\partial X_2}{\partial Y} \cdot \frac{\partial Y}{\partial P_Y} \cdot \frac{\partial C_Y}{\partial P_1}\ . \tag{18}$$

By multiplying both sides of the equation by P_1/X_2 and rearranging, we get

$$\frac{dX_2}{dP_1} \cdot \frac{P_1}{X_2} = \left(\frac{\partial X_2}{\partial P_1} \cdot \frac{P_1}{X_2}\right)_{X_3 = X_3^0,\ Y = Y_0} + \left(\frac{\partial X_2}{\partial X_3} \cdot \frac{\partial X_3}{\partial P_1} \cdot \frac{P_1}{X_2} \cdot \frac{X_3}{X_3}\right)_{Y = Y_0} + \frac{\partial X_2}{\partial Y} \cdot \frac{\partial Y}{\partial P_Y} \cdot \frac{\partial C_Y}{\partial P_1} \cdot \frac{P_1}{X_2} \cdot \frac{P_Y}{Y} \cdot \frac{Y}{P_Y}\ . \tag{19}$$

In equation (19), the following equalities hold:

$$\left(\frac{\partial X_3}{\partial P_1}\cdot\frac{P_1}{X_3}\right)_{Y=Y_o} = E_{31}\,|_{Y=Y_o},$$

$$\left(\frac{\partial X_2}{\partial X_3}\cdot\frac{X_3}{X_2}\right)_{Y=Y_o} = a_{23}\,|_{Y=Y_o},$$

$$\frac{\partial X_2}{\partial Y}\cdot\frac{Y}{X_2} = 1 \quad \text{from constant returns to scale,}$$

$$\frac{\partial Y}{\partial P_Y}\cdot\frac{P_Y}{Y} = E_{YY}, \quad \text{and}$$

$$\frac{\partial C_Y}{\partial P_1}\cdot\frac{P_1}{P_Y} \doteq \frac{X_1}{Y}\cdot\frac{P_1}{P_Y} = S_1.$$

So equation (19) is simplified into

$$E_{21} = E_{21}\,|_{X_3=X_3^o,\,Y=Y_o} + (a_{23}\cdot E_{31})|_{Y=Y_o} + S_1\cdot E_{YY}, \tag{20}$$

where a_{23} is the elasticity of X_2 with respect to X_3 at $Y = Y_o$, and S_1 is the cost share of X_1 in the total cost of Y. Furthermore,

$$E_{21}\,|_{X_3=X_3^o,\,Y=Y_o} = \left(\frac{\partial X_2}{\partial P_1}\cdot\frac{P_1}{X_2}\right)_{X_3=X_3^o,\,Y=Y_o} = \left(C_{21}^3\cdot\frac{P_1}{C_2^3}\right)_{X_3=X_3^o,\,Y=Y_o}$$

$$= \left(\frac{P_1\cdot C_1^3}{C^3}\cdot\frac{C^3\cdot C_{21}^3}{C_2^3\cdot C_1^3}\right)_{X_3=X_3^o,\,Y=Y_o} = S_{1X_3}\cdot\sigma_{21}^D, \tag{21}$$

where C^3 is the total cost of X_3, $C_i^3 = \partial C^3/\partial p_i = X_i$, $C_{ij}^3 = \partial^2C^3/\partial p_i\cdot\partial p_j$, S_{1X_3} is the cost share of X_1 in the cost of $X_3 = H(X_1,X_2)$ at the given output level, and σ_{21}^D is the direct partial elasticity of substitution between X_2 and X_1.[8]

Similarly,

$$E_{31}\,|_{Y=Y_o} = \left(\frac{\partial X_3}{\partial P_1}\cdot\frac{P_1}{X_3}\right)_{Y=Y_o} = S_1\cdot\sigma_{31}^A, \tag{22}$$

where S_1 is the cost share of X_1 in the cost of Y and σ_{31}^A is the Allen partial elasticity of substitution between X_3 and X_1. So equation (20) can be written as

8. Direct elasticity of substitution deals with only the two factors directly involved keeping other factors contant. For further discussion of DES, refer McFadden (1963).

$$E_{21} = S_{1X_3} \cdot \sigma_{21}^{D} + S_1 \cdot a_{23} \cdot \sigma_{31}^{A} + S_1 \cdot E_{YY}$$
$$= S_1 \, [(\frac{1}{S_{X_3}} \; \sigma_{21}^{D} + a_{23} \cdot \sigma_{31}^{A}) + E_{YY}] , \qquad (23)$$

where $S_{X_3} = S_1 / S_{1X_3}$

From equation (23) we see that the output-variable cross price elasticity is affected by the following three effects: the substitution effect, the third-factor effect, and the output effect. If the output is held constant,

$$E_{21} \big|_{Y = Y_o} = S_1 \; (\frac{1}{S_{X_3}} \; \sigma_{21}^{D} + a_{23} \cdot \sigma_{31}^{A}). \qquad (24)$$

The bracketed term is in fact the Allen partial elasticity of substitution σ_{21}^{A} when the production function $Y = F(X_1, X_2, X_3)$ is not characterized by the separability of X_1 and X_2 from X_3. In general,[9]

$$\sigma_{ij}^{A} = \frac{1}{S_{X_k}} \; \sigma_{ij}^{D} + a_{ik} \cdot \sigma_{kj}^{A} , \qquad \begin{array}{l} i, j = 1, 2, 3 \\ i \neq j \\ k = 1, 2, 3 \text{ except } i \text{ and } j \, . \end{array} \qquad (25)$$

From equation (25) we see that the Allen partial elasticity of substitution between X_i and X_j is composed of the Allen partial elasticity of substitution between X_j and X_k as well as on the direct partial elasticity of substitution between X_i and X_j. If the third-factor effect has the same positive sign as the substitution effect (which is always positive), the Allen partial elasticity of substitution between X_i and X_j will be greater than the direct elasticity of substitution. But if the negative third-factor effect dominates the positive substitution effect, the Allen partial elasticity of substitution will be negative. So the magnitude of the Allen partial elasticity of substitution and the direct elasticity of substitution cannot be compared *a priori* when the separability condition is not satisfied.

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9 The justification of the equation (25) can be proved utilizing the Jacobi's theorem on determinants which is as follows.

$$F_{ij} F_{kk} \quad F_{ik} F_{kj} = F \cdot F_{kk.ij} .$$

Application of the Jacobi's theorem on the Allen partial elasticity of substitution will give

$$\sigma_{ij}^{A} = \frac{\sum_{V=1}^{n} f_V X_V}{X_i X_j} \cdot \frac{F_{ij}}{F} \quad \frac{\sum_{V=1}^{n} f_V X_V}{X_i X_j} \{ \frac{F_{kk,ij}}{F_{kk}} + \frac{F_{kj}}{F} \cdot \frac{F_{ik}}{F_{kk}} \}$$

$$= \frac{\sum_{V=1}^{n} f_V X_V}{\sum_{\substack{V=1 \\ V \neq 1}}^{n} f_V X_V} (\frac{\sum_{\substack{V=1 \\ V \neq k}}^{n} f_V X_V}{X_i X_j} \cdot \frac{F_{kk,ij}}{F_{kk}}) + (\frac{\sum_{V=1}^{n} f_V X_V}{X_j X_k} \cdot \frac{F_{kj}}{F}) \; (\frac{\frac{F_{ik}}{F}}{\frac{F_{kk}}{F}} \cdot \frac{X_k}{X_i}) = \frac{1}{S_{X_k}} \; \sigma_{ij}^{D} + \sigma_{kj}^{A} \cdot a_{ik} .$$

By combining equations (23) and (25) we get

$$E_{ij} = S_j\ (\sigma_{ij}^A + E_{YY}), \tag{26}$$

which looks identical with equation (17). The difference between equations (17) and (26) lies on the components of the Allen partial elasticities of substitution. When there are more than three factors of production, the components of Allen partial elasticities of substitution will be different depending on the separability conditions assumed. This assertion is evidenced by equations (15) and (25), where the Allen partial elasticities of substitution are formulated with and without weak separability conditions imposed, respectively.

5. Concluding Remarks

In this paper we combined together the Allen-Robinson formula for the elasticity of derived demand and the Sato's formula for the relation between Allen partial elasticities of substitution and inter- and intra-group elasticities of substitution following step-by-step the firm's response to a factor price change. In the process it is discovered that the Sato's formula can be derived only if the weak separability condition is assumed.

Then we abandoned the separability assumption to see how the Sato's formula should be modified for it to be a general one. This paper shows that without the separability assumption the Allen partial elasticity of substitution is the combination of direct partial elasticity of substitution which is analogous to intra-group elasticity of substitution and the third-factor effect. The sign of the third-factor effect depends on the sign of the Allen partial elasticities of substitution between the factor whose price has changed and the third factor. Thus unlike the expansion elasticity whose sign is always negative, the sign of the third-factor effect cannot be decided *a priori*.

One of the implications of this finding is that Berndt-Wood's attempt to reconcile the Energy-Capital Complementarity Controversy with the Sato's formula loses its ground if the separability condition is not empirically tested and accepted beforehand.

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