The Choice of Optimal Basket in Nominal Exchange Rate Peg

Simon Joong-Woong Pak*

Young On Kim**

I. Introduction

Because of high real costs associated with volatility of exchange rate changes for the past decade, many countries, especially small economies, have sought to fix their exchange rates to some relatively stable standard. However, it is not immediately clear against which currency or currency composite they should peg the domestic currency.

Lipschitz (1979) shows that the standard to which the domestic currency is pegged influences target variables of policy makers' concern such as income distribution between capital and labor, internal terms of trade, resource allocations between tradable and non-tradable sectors, and balance of payments. This implies that the choice of currency composite should depend upon policy makers' objectives. The problem of determining an appropriate currency composite for a given economic objective can conveniently be formulated as an optimization problem which determines an optimal weighted currency composite.

In the paper by Flanders and Helpman (1979), an optimal weighted currency composite (OWCC) is obtained by setting up a model in which the variance of an economic target variable is minimized subject to a basket peg and a predetermined desired chage in the target variable. The target variables they used are trade balance and welfare. The OWCC for one target variable differs from that for the other target variable.

Another approach suggested by Lipschitz (1979) is to choose an OWCC by minimizing fluctuation of a real effective exchange rate. Lipschitz & Sundararajan (1980) obtain an OWCC explicitly by formulating and solv-

* Florida International University, U.S.A.

** New York University, U.S.A.

This paper was presented at the International Convention of Korean Economists, Seoul, 20-25 Aug., 1984.

ing a constrained optimization problem where the objective is to minimize the variance of a real exchange rate index subject to a nominal exchange rate pegging, non-negativity conditions for currency weights, and an upper and a lower limits on the movement of the real exchange rate.

One objective of this paper is to generalize the method of Flanders and Helpman by specifying the economic target variable as a funciton of not only exchange rates as in Flanders & Helpman but also other variables such as relative prices between home country and trading partners and incomes of trading partners. This will enable us to show that the OWCC obtained by Flanders and Helpman and that by Lipschitz and Sundararajan are special cases of the OWCC of the general model. In addition, it will be shown that covariances between exchange rate and other variables such as income of trading partners play a role in determining the OWCC in the generalized model while this role is not explicitly treated in the two previous papers.

The approach described above is applicable to a situation where the policy makers' concern is limited to reducing the fluctuation of a single economic target variable caused by exchange rate fluctuation. If, however, policy makers want to construct a currency basket so that variances of more than one target variables a new approach is needed. The second objective of this paper is to develop a framework in which an OWCC with multiple targets can be obtained by specifying the objective function as a weighted sum of variances of all the target variables. Such an OWCC is found to be a weighted average of the single target OWCC's obtained for each of the target variables, where the weight for a single target OWCC is related to the importance of the variance of the target variable in the overall objective.

Section II deals with the determination of an OWCC when the objective of the policymakers is to minimize the variance of a single target. In section III the relationship between minimizing the variance of a real exchange rate index and minimizing the variances of the target variable is examined, and the conditions under which the two methods are equivalent to each other are identified. Section IV extends the method to the case with multiple targets in obtaining an OWCC. Section V summarizes the results.

II. Optimal Basket with a Single Target

This section presents the problem of determining an optimal weighted currency composite (OWCC) as an optimization problem where the objective is to minimize the variance of a target economic variable. Lipschitz and Sundararajan (1980) obtained an OWCC by minimizing the variance of a real exchange rate index. The approach in this section takes the method of Lipschitz and Sundararajan one step further in that an optimal weighted currency composite is always obtainable even when the objective of the policymakers is something other than the real exchange rate. This approach is also a generalization of Flanders and Helpman (1979) in that the target variable is a function of not only exchange rates but also any other variables.

Following the notations of Lipschitz and Sundararajan, assume that a small country trades with n partner countries, i = 1, . . . , n. Further assume that the pound sterling (i = 1) be the numeraire currency and the domestic currency be called the rupee.

e'_{it} = pounds per unit of the i^{th} currency

e'_t = pounds per ruppee

$e_{it} = e'_t / e'_{it}$ = i^{th} currency units per rupee.

The subscript t refers to the time period, and the subscript 0 refers to the base date of indices.

If the rupee is pegged to a log-linear basket of n currencies, with weights w_i, i = 1, . . . ,n,

$$\ln\left(\frac{e'_t}{e'_o}\right) = \sum_{i=1}^{n} w_i \, \ell n \left(\frac{e'_{it}}{e'_{io}}\right), \tag{1}$$

then the problem of determining an OWCC is to determine the weights (w_i 's) to optimize an objective function of the authorities.

Suppose the objective of the authorities is to minimize the variance of a certain variable X_t which depends upon the exchange rates e_{it}, i = 1 to n:

$$X_t = X(e_{1t}, \ldots e_{nt} ; Z_{1t}, \ldots Z_{mt}) \tag{2}$$

where Z_{it} 's, j = 1, . . . , m, are variables other than e_{it} 's such as domestic and foreign price levels and income levels during the time period t. To make the problem algebraically manageable, it is assumed that exchange rates (e_{it}) and Z_{jt} fluctuate around e_{io} and Z_{jo} respectively by small amounts, so that X_t can be approximated by a Taylor series expansion. The first order approximation is:

$$X_t = X_o + \sum_{i=1}^{n} \frac{\partial X}{\partial e_{it}} (e_{it} - e_{io}) + \sum_{j=1}^{m} \frac{\partial X}{\partial z_{jt}} (z_{jt} - z_{jo}) \tag{3}$$

The changes in e_{it} and Z_{jt} are further approximated using $\ell n\,(1 + \varepsilon) \simeq \varepsilon$ for a small ε :

$$e_{it} - e_{io} \simeq e_{io} \, \ell n \left(\frac{e_{it}}{e_{io}}\right) \tag{4a}$$

$$Z_{jt}-Z_{jo} \simeq Z_{jo}\ \ell n\ (\frac{Z_{jt}}{Z_{jo}}) \tag{4b}$$

$$X_t-X_o \simeq X_o\ \ell n\ (\frac{X_t}{X_o}) \tag{4c}$$

Utilizing eqs. (4a), (4b), and (4c), the expression for X_t is approximated as:

$$\ell n\ (\frac{X_t}{X_o}) \simeq \sum_{i=1}^{n} \eta(X: e_i)\ \ell n\ (\frac{e_{it}}{e_{io}}) + \sum_{j=1}^{m} \eta(X: Z_j)\ \ell n\ (\frac{Z_{jt}}{Z_{jo}}) \tag{5}$$

where η (X : e_i) and η (X : Z_j) are elasticities of X_t with respect to e_{it} and Z_{jt}, respectively:

$$\eta\ (X{:}e_i) \equiv [\frac{e_{it}}{X_t} \cdot \frac{\partial X_t}{\partial e_{it}}]_{t=0} \tag{6a}$$

$$\eta(X{:}\ Z_j) \equiv [\frac{Z_{jt}}{X_t} \cdot \frac{\partial X_t}{\partial Z_t}]_{t=0} \tag{6b}$$

Noting $e_{it} = e_t'/e_{it}'$, the expression (5) can be rewritten as:

$$\ell n\ (\frac{X_t}{X_o}) = \eta(X)\ \ell n\ (\frac{e_t'}{e_o'}) - \sum_{i=1}^{n} \eta(X{:}e_i)\ \ell n\ \frac{e_{it}'}{e_{io}'} + \sum_{j=1}^{m} \eta(X{:}Z_j)\ \ell n \frac{Z_{jt}}{Z_{jo}} \tag{7}$$

$$\text{where:} \quad \eta(X) \equiv \sum_{i=1}^{n} \eta(X{:}\ e_i)\ . \tag{8}$$

Substituting eq. (1) for the term $\ell n\ (e_t'/e_0')$ in eq. (7), we obtain:

$$\ell n\ (\frac{X_t}{X_o}) = \eta(X) \sum_{i=1}^{n} [w_i - \eta(X{:}\ e_i)/\eta(X)]\ \ell n \frac{e_{it}'}{e_{io}'} + \sum_{j=1}^{m} \eta(X{:}\ Z_j)\ \ell n \frac{Z_{jt}}{Z_{jo}}\ . \tag{9}$$

Noting that for a small change in X_t:

$$\text{var}\ (X_t) \equiv E(X_t - X_o)^2 \simeq X_o^2\ E\ [\ell n\ (\frac{X_t}{X_o})]^2 \tag{10}$$

the optimization problem can be stated as:

$$\underset{\{w_i\ :\ i=1\sim n\}}{\text{minimize}} \quad E\ [\ell n\ (\frac{X_t}{X_o})]^2 \tag{11}$$

subject to:

$$\sum_{i=1}^{n} w_i = 1 \tag{12}$$

where $\ell n\left(\frac{X_t}{X_o}\right)$ is defined by eq. (9).

As in Lipschitz and Sundararajan, additional constraints may be added such as:

(i) non-negativity of w_i's for all $i = 1 \sim n$ and

(i) imposing upper and lower bounds on the nomial exchange rate.

This is a quadratic programing problem very similar to that in Lipschitz and Sundararajan (1980). Details of the general solution are given in the appendix as a special case of multiple targets. An interior solution, if it exists, is presented below.

$$w_s = \frac{1}{\eta(X)}\left[\eta\,(X\colon e_s) - \sum_{j=1}^{m} \eta(X\colon Z_j)\;\; c(e'_s Z_j)/v(e'_s)\right] \tag{13a}$$

for $s = 2 \sim n$

$$w_1 = 1 - \sum_{s=2}^{n} w_s \tag{13b}$$

where:

$$c(e'_s\, Z_j) \equiv E\left[\left\{\ell n\left(\frac{e'_{st}}{e'_{so}}\right)\right\}\left\{\ell n\left(\frac{Z_{jt}}{Z_{jo}}\right)\right\}\right. \;;\; s=2, \ldots, n, j = 1, \ldots, m \tag{14a}$$

$$v(e'_s) \equiv E\left[\ell n\left(\frac{e'_{st}}{e'_{so}}\right)\right]^2 \tag{14b}$$

In obtaining the solution above, it is assumed that:

$$c(e'_i\, e'_j) \equiv E\left[\left\{\ell n\left(\frac{e'_{jt}}{e'_{io}}\right)\right\}\left\{\ell n\left(\frac{e'_{jt}}{e'_{jo}}\right)\right\}\right] = 0 \text{ for } i \neq j \tag{15}$$ [1]

For an illustrative purpose, the case of trade balance is considered as an example. Trade balance (TB_t) is the total net export. Denoting exports to and imports from the trading partner i by X_{it} and M_{it} respectively. TB_t can be expressed as:

$$TB_t = \sum_{i=1}^{n} (X_{it} - M_{it}) \tag{16}$$

and

$$X_{it} = X_{it}\,(Q_{it}, Y_{it}), \quad \frac{\partial X_{it}}{\partial Q_{it}} < 0, \; \frac{\partial X_{it}}{\partial Y_{it}} > 0 \tag{17}$$

1) When this assumption is relaxed, the weight W_s corresponding to eq. 13a can be obtained in matrix form.

$$M_{it} = M_{it}\,(Q_{it}, Y_t), \quad \frac{\partial M_{it}}{\partial Q_{it}} > 0, \frac{\partial M_{it}}{\partial Y_t} > 0 \tag{18}$$

where:

$Q_{it} \equiv \frac{P_t\, e_{it}}{P_{it}}$, P_t and P_{it} are price levels in the home country and i^{th} country, respectively; and Y_{it} and Y_t denote the levels of income of the i^{th} trading partner and of domestic country respectively.

For the sake of notational convenience, let

$$\frac{P_t}{P_{it}} \equiv R_{it}\,.$$

Then $Q_{it} = R_{it} \cdot e_{it}$

From equations (16), (17), and (18), it can be shown easily that

$\eta(TB : e_s) = \eta(TB : R_s) = \eta(TB : Q_s)$

and

$$\eta(TB : Q_s) = \frac{X_{so}\,\eta(X_s : Q_s) - M_{so}\,\eta(M_s : Q_s)}{\sum_{i=1}^{n} (X_{io} - M_{io})} \qquad \text{for } s=1, \ldots, n,$$

where $\eta\,(X_s : Q_s)$ and $\eta\,(M_s : Q_s)$ are price elasticities of export and import respectively for the i^{th} country.

Also note that:

$\eta\,(TB : Y_s) = \eta(X_s : Y_s)$,

$\eta\,(TB : Y) = -\eta(M_s : Y)$,

and

$$\eta(TB) = \sum_{j=1}^{n} \eta(TB : Q_j\,) = \frac{\sum_{j=1}^{n} [X_{jo}\,\eta(X_j : Q_j) - M_{jo}\,\eta(M_j : Q_j)]}{\sum_{i=1}^{n} (X_{io} - M_{io})}$$

Assuming $c(e_s R_j) = 0$, $c(e_s' Y_j) = 0$ for $s = j$, and $c(e_s'\,Y) = 0$, and applying eq. (13a), we obtain:

$$w_s = \frac{\eta(TB:Q_s)}{\eta(TB)}\left[1 - \frac{c(e_s'\,R_s)}{V(e_s')}\right] - \frac{\eta(X_s : Y_s)}{\eta(TB)}\,\frac{c(e_s'\,Y_s)}{V(e'_s)}, \tag{19}$$

for $s=2, \ldots, n$.

For a moment, terms including covariances will be ignored, or covariances are assumed to be zero even though it is very unlikely. Then w_s can be expressed using elasticity weights, i.e.,

$$W_s = \frac{X_{so}\ \eta(X_s:Q_s) - M_{so}\ \eta(M_s:Q_s)}{\sum_{i=1}^{n}[X_{io}\ \eta(X_i:Q_i) - M_{io}\ \eta(M_i:Q_i)]} \qquad s=2, \ldots, n \qquad (20)$$

We can consider special cases by restricting the elasticities further. If all the elasticities are unit elastic, then W_s become familiar tradevolumn weights, for $s = 1, \ldots, n$. Assuming import price elasticities for all partner countries are perfectly inelastic, w_s are export weights. An example may be an economy whose main import item is crude oil which is not produced domestically at all. Import weights are optimal when export price elasticities are prefectly inelastic.

Naive elasticity weights, however, are not in general optimal wieghts in a world of generalized floating with inflation. the above example suggests that authorities should have a good knowledge of the structure of the economy as well as the degree of fluctuations of relevant variables in order to get an OWCC.

III. Minimization of the variance of a real exchange rate index

This section discusses a real exchange rate index for a given target variable and examines the relationship between minimization of the variance of a target variable and that of the corresponding real exchange rate index.

A real exchange rate index with respect to a target variable may be constructed by combining a nominal effective exchange rate index and a relative price index. [Lipschitz (1979), Lanyi and Suss (1982), Maciejewski (1983)]. Rhomberg (1976) defines a nominal effective exchange rate with respect to the trade balance variable as:

> "In terms of the trade balance objective, the change in the effective exchange rate may be defined as the notional *uniform* proportionate change in the price of the home currency in terms of foreign currencies that would have the same effect on the home country's trade balance as the set of *actual* changes in these prices. To calculate this notional uniform exchange rate change, the actual change in the exchange rate vis-a-vis each foreign currency must be weighted by the effect on the home country's trade balance of an isolated change in the price of that currency in a given proportion, say, by 1 percent."

Rhomberg's definition of the nominal effective exchange rate with respect to the trade balance variable can be generalized to any other target variables by substituting the words "target variable" for the "trade balance" in the above quote.

To be more concrete, denote the uniform proportionate change in the price of the rupee by d (IN_t/IN_0), where IN_t is the nominal effective exchange rate index. Then the effect of nonimal exchange rate changes on X_t are:

$$dX_t = \sum_{i=1}^{n} \left[\frac{\partial X_t}{\partial e_{it}}\right]_{t=0} \cdot de_{it}.$$

Setting $de_{it}/e_{io} = d(IN_t)/IN_0$ for $i = 1 \sim n$, the expression above is rewritten as:

$$\ell n \left(\frac{X_t}{X_o}\right) = \sum_{i=1}^{n} \eta(X : e_i) \, \ell n \left(\frac{IN_t}{IN_o}\right).$$

That is:

$$\ell n \left(\frac{IN_t}{IN_o}\right) = \frac{1}{\eta(X)} \, \ell n \left(\frac{X_t}{X_o}\right). \tag{21}$$

The effects of the actual changes in the price of the rupee are:

$$\ell n \left(\frac{X_t}{X_o}\right) = \sum_{i=1}^{n} \eta(X : e_i) \, \ell n \left(\frac{e_{it}}{e_{io}}\right). \tag{22}$$

Substituting (22) for $\ln(X_t/X_0)$ in eq. (21), we obtain an expression for a nominal effective exchagne rate index for the target variable X_t as:

$$\ell n \left(\frac{IN_t}{IN_o}\right) = \sum_{i=1}^{n} [\eta(X : e_i) / \eta(X)] \, \ell n \left(\frac{e_{it}}{e_{io}}\right). \tag{23}$$

Using the same weights as in the nominal effective exchange rate, a relative price index (IP_t) for the variable X_t can be constructed as:

$$\ell n \left(\frac{IP_t}{IP_o}\right) = \sum_{i=1}^{n} [\eta(X : e_i)/\eta(X)] \, \ell n \left(\frac{P_t}{P_{it}} / \frac{P_o}{P_{io}}\right), \tag{24}$$

where P_t and P_{it} are appropriate price levels in the home country and the i^{th} foreign country, respectively. Then the real exchange rate index (IR_t) for the target variable X_t may be defind as:

$$\begin{aligned} \ell n(IR_t/IR_o) &\equiv \ell n(IN_t/IN_o) + \ell n(IP_t/IP_o) \\ &= \sum_{i=1}^{n} [\eta(X : e_i)/\eta(X)] \, \ell n\left(\frac{e_{it} \, P_t \, P_{io}}{e_{io} \, P_o \, P_{it}}\right). \end{aligned} \tag{25}$$

An OWCC shich minimizes the variance of the real exchange rate index $\ln(IR_t/IR_0)$, is obtained in Lipschitz and Sundararajan (1980).

Comparing eq. (25) with eq. (5), one can easily identify the conditions under which the OWCC obtained through the minimization of the variance of $\ln(X_t/X_0)$ is equivalent to the OWCC through the minimization of the variance of $\ln(IR_t/IR_0)$ They are:

(i) $Z_{it} \equiv P_t/P_{it}$ for i=1~n ; (26)

(ii) $\eta(X:e_i) = \eta(X:Z_i)$ for i=1~n; and (27)

(iii) $\ell n\,(e_{it}/e_{io})$ and $\ell n(Z_{jt}/Z_{jo})$ are not correlated with each other for any i and j > n if m > n. (28)

Target variables of the following type satisfy all three conditions above:

$$X_t = X_t\,(\frac{e_{it}\,P_t}{P_t}\,,\frac{e_{2t}\,P_t}{P_{2t}}\,,\ldots,\frac{e_{nt}\,P_t}{P_{nt}}\,,\,Z_{(n+1)t},\,Z_{(n+2)t},\ldots,Z_{mt}) \tag{29}$$

where Z_{jt}'s for $j \geqslant (n+1)$ are not correlated with any of the exchange rate e_{it} for $i = 1 \sim n$.

The remaining section examines whether the two types of OWCC are equilvalent to each other using three specific target variables from Lipschitz (1979), which are trade balance, internal terms of trade, and income distribution.

When the target variable is the trade balance, the OWCC's obtained by the two methods are not equivalent to each other. From the example in the previous section the trade balance (TB_t) can be expressed in general form as:

$$TB_t = \phi\,(Y_t;\frac{e_{it}\,P_t}{P_{it}},\,Y_{it},\ i=1\sim n)\,. \tag{30}$$

Since the real income (Y_{it}) and the currency value of a country ($e_{it}\,P_t/P_{it}$) are not in general independent from each other, the trade balance expressed in eq. (30) does not satisfy the condition (28). This can also be seen from the equation (19). The OWCC obtained by Lipschitz and Sundararajan (1980) includes only the term

$$\frac{\eta(TB:Q_s)}{\eta(TB)}\,[1-\frac{c(e_s'\,R_s)}{V(e_s')}]\,.$$

Therefore, the OWCC obtained by minimizing the variance of the real exchange rate index for the trade balance will not minimize the variance of the trade balance.

Internal terms of trade (IT_t) may be defined as the ratio of the price index of non-traded goods (P_t^N) to the price index of raded goods (P_t^N) where the price index of traded goods may be dfined as:

$$P_t^T = \prod_{i=1}^{n}\,(\frac{P_{it}}{e_{it}})^{\alpha_i} \tag{31a}$$

$$\sum_{i=1}^{n} \alpha_i = 1 . \tag{31b}$$

Then

$$IT_t = P_t^N / [\prod_{i=1}^{n} (P_{it}/e_{it})^{\alpha_i}] = \prod_{i=1}^{n} [\frac{e_{it} P_t^N}{P_{it}}]^{\alpha_i} . \tag{32}$$

Since IT_t expressed in eq. (32) satisfies all the three conditions, (26), (27) and(28), the OWCC obtained by minimizing the variance of $\ell n(IT_t/IT_0)$ should be same as that obtained by minimizing the variance of the real exchange rate index for IT_t, whic is:

$$\ell n(\frac{IR_t}{IR_o}) = \sum_{i=1}^{n} \alpha_i \ell n (\frac{e_{it} P_t^N P_{io}}{e_{io} P_o^N P_{it}}) . \tag{33}$$

When the target variable is a measure of income distribution, the OWCC's obtained by the two methods are equivalent to each other as in the case of internal terms of trade. As in Lipschitz (1979), the economy is divided into three sectors: the exporting (x), import-competing (mc), and non-traded goods producing (N). The output of each is denoted by Q, the price of this output in domestic currency by P. One measure of income distribution (ID_t) may be defined as the ratio of the capitalists' real income (RR) to the workers' real income (WR). The capitalists' real income is determined by the value of output less the wage bill deflated by the consummer price index (P_c) and the workers' real income is the exogenously fixed wage rate (W) times employment (L) deflated by the consumer price index. The capitalists' real income is expressed as:

$$RR_t = [P_{xt} Q_{xt} + P_{mt} Q_{mct} + P_t^N Q_{nt} - w_t L_t] / P_{ct}$$

where P_{mt} is the price index for imported goods.
The workers' real income is:

$$WR_t = \frac{W_t L_t}{P_{ct}}$$

Thus, the relative income (ID_t) is expressed as

$$ID_t = \frac{P_{xt} Q_{xt} + P_{mt} Q_{mct} + P_t^N Q_{Nt} - W_t L_t}{W_t L_t} \tag{34}$$

The export and import price indices may be defined as:

$$P_{xt} \equiv \prod_{i=1}^{n} (P_{it}/e_{it})^{\alpha_{xi}} \tag{35}$$

$$P_{mt} \equiv \prod_{i=1}^{n} (P_{it}/e_{it})^{\alpha_{mi}} \tag{36}$$

where $$\sum_{i=1}^{n} \alpha_{xi} = \sum_{i=1}^{n} \alpha_{mi} = 1 \tag{37}$$

If P_t^N an w_t are treated as exogenously determined, and if the outputs and labor (Q_{xt}, Q_{mct}, Q_{Nt}, and L_t) are either treated as exogenously determined or assumed to depend upon exchange rates in the form of $(\frac{e_{it}\, P_t^N}{P_{it}})$ only, then the income distribution measure (ID_t) satisfies the three conditions (26), (27) and (28). Thus, as in internal terms of trade, the OWCC obtained by minimizing the variance of the real exchange rate index should be same as that obtained by minimizing the variance of $\ln(ID_t/ID_o)$.

IV. Optimal Basket with Multiple Targets

When there are more than one target variables of policy makers' concern, an OWCC obtained by minimizing the variance of any particular variable does not necessarily minimize the variances of all other target variables. A reasonable approach to this type of problem is to minimize a weighted sum of the variances of all the target variables where the weights are assigned subjectively by the policy makers according to the importance of each target variable. This section solves for an OWCC when the objective of policy makers is to minimize a weighted sum of the variances of all the target variables. It is assumed that there are I target variables denoted by $X_t^i(e_{1t}, \ldots, e_{nt}; Z_{1t}, \ldots, Z_{mt})$, $i = 1 \sim I$. The objective function of the policy makers may be expressed as:

$$Q_t \equiv \sum_{i=1}^{n} a_i\, E[\ln(\frac{X_t^i}{X_o^1})]^2 \tag{38}$$

where a_i is the weight given to the i^{th} target variable X_t^i.

Using eq. (5), we obtain from eq. (38)

$$Q_t = \sum_{i=1}^{n} a_i\, E[\sum_{j=1}^{n} \eta(X^i{:}e_j) \ln(\frac{e_{jt}}{e_{jo}}) + \sum_{j=1}^{n} \eta(X^i{:}Z_j) \ln(\frac{Z_{jt}}{Z_{jo}})]^2 \tag{39}$$

Through the same procedure used in obtaining eq. (9) from eq. (5), the expression for Q_t(eq. (39)) may be transformed into:

$$Q_t = \sum_{i=1}^{n} a_i \, [\eta(X^i)]^2 \; E[\sum_{j=1}^{n} \{w_j - \eta\,(X^i{:}e_j)/\eta(X^i)\} \; \ell n\,(\frac{e_{it}'}{e_{jo}'}) + \sum_{j=1}^{m} \{\eta(X^i{:}Z_j)/\eta(X^i)\} \; \ell n\,(\frac{Z_{jt}}{Z_{jo}})]^2 \tag{40}$$

where $\quad \eta(X^i) \equiv \sum_{j=1}^{n} \eta(X^i{:}e_j), \; i=1\sim I$. $\qquad$ (41)

The task of determining an OWCC is to find the weights w_i 's, $i=1\sim n$, in the following optimization problem:

$$\underset{\{w_i, \ldots w_n\}}{\text{minimize}} \quad Q_t = \Sigma \; a_i \; E[\ell n\,(\frac{X_t^i}{X_o^i})]^2$$

subject to the condition (1) and (12).
This is a quadratic programing problem. The case where an interior solution exist is presented below. The details of the general solution is in the appendix. Assuming that:

$$E\,[(\ell n\,(\frac{e_{it}'}{e_{io}'})\,(\ell n \frac{e_{jt}'}{e_{jo}'})\,] = 0 \text{ for } i \neq j \tag{42}$$

one can obtain the wieghts for an OWCC as:

$$w_s = \sum_{i=1}^{n} A_i \, [\frac{\eta(X^i{:}e_s)}{\eta(X^i)} - \sum_{j=1}^{m} \frac{\eta(X^i{:}Z_j)}{\eta(X^i)} \; \frac{c(e_s' \, z_j)}{V(e_s')}]; \; s=2\sim n \tag{43a}$$

$$w_i = 1 - \sum_{s=2}^{n} w_s \tag{43b}$$

where: $\quad A_i \equiv a_i [\eta(X^i)]^2 \,/ \sum_{j=1}^{I} a_j [\eta(X^j)]^2.$ $\qquad$ (44)

By comparing the OWCC for a single target, eqs. (13a,b), with that for the multiple-targets, eqs. (43a,b), one can conclude that the multiple target OWCC is an average of the single-target OWCC's, each weighted by the importance of the individual target (A_i),

V. Conclusion

An optimal weighted currency composite (OWCC) for a nominal peg is obtained by specifying the objective of policy makers as minimizing a weighted sum of variances of target variables. In the case of a single target variable, it has been shown that the OWCC obtained by minimizing the variance of a real exchange rate index defined for the trarget variable is not always identical to the OWCC obtained by minimizing the variance of the taraget variable directly. When the single target variable is the trade balance, the OWCC obtained by one method is different from the OWCC obtained by the other method, while the two OWCC's are identical to each other when the single target variable is income distribution or the internal terms of trade.

In the case of multiple target variables, the OWCC is found to be a weighted average of the single-target OWCC's obtained for each of the target variables, where the weight for a single-target OWCC is related to the importance of the variance of the target variable in the overall objective.

Appendix

Optimal Basket with Multiple Targets

The task of authorities is to choose a set of weights $w_i \ldots,\ w_n$ that minimizes the objective function in the equations (40).

$$Q_t = \sum_{i=1}^{I} a_i \eta^2 (X^i)\, E\{ \sum_{j=1}^{n} [w_j - \eta(X^i{:}e_j) / \eta(X^i)]\, \ell n \left(\frac{e'_{jt}}{e'_{jo}}\right) + \sum_{k=1}^{m} [\eta(X^i{:}Z_k)/\eta(X^i)]\, \ell n \left(\frac{Z_{kt}}{Z_{ko}}\right)\}^2$$

subject to constraints

$$B_\ell \leqslant \sum_{j=1} w_j \bar{e}'_j \leqslant B_u$$

$$\sum_{j=1} w_j = 1$$

and

$$w_j \geqslant 0,\ j=1, \ldots, n$$

where

$$\bar{e}'_j \equiv E(\ell n \frac{e'_{jt}}{e'_{jo}}).$$

Since the concern is only with the set of bilateral relations between rupee and the currency of each partner country, convariances among partner countries' exchange rates will be ignored, i.e., it will be conveniently that

$$\mathrm{cov}\, (\ell n \frac{e'_{jt}}{e'_{jo}},\ \ell n \frac{e'_{kt}}{e'_{ko}}) = 0 \text{ for } j \neq k$$

The objective functio Q_t can be rewritten as

$$Q_t = \sum_{i=1}^{I} a_i \eta^2 (X^i)\, [\sum_{j=2}^{n} \{w_j - \eta(X^i{:}e_j) / \eta(X^i)]^2\ V(e'_j)$$
$$+ 2 \sum_{j=2}^{n} \sum_{k=1}^{m} [w_j - \eta(X^i{:}e_j) / \eta(X^i)]\ [\eta(X^i{:}Z_k) / \eta(X^i)]\ c(e'_j\ Z_k)$$
$$+ \sum_{k=1}^{m} \sum_{\ell=1}^{m} \frac{\eta(X^i{:}Z_k)\, \eta(X^i{:}Z_\ell)}{\eta^2 (X^i)}\, c(Z_k\ Z_\ell)\} \quad \ldots\ldots (A.\ 1)$$

where

$$V(e_j') \equiv E[\ln \frac{e_{jt}'}{e_{jo}'}]^2$$

$$c(e_j' Z_k) \equiv E[\ln \frac{e_{jt}'}{e_{jo}'} . \ln \frac{Z_{kt}}{Z_{ko}}]$$

$$c(Z_k Z_\ell) \equiv E[\ln \frac{Z_{kt}}{Z_{ko}} . \ln \frac{Z_{\ell t}}{Z_{\ell o}}], \qquad j=2, \ldots, n \quad k, \ell = 1, \ldots, m.$$

Note $V(e_1') = 0$, $c(e_1 Z_k) = 0$ for $k = 1, \ldots, m$, and $e_1' = 0$.
The Lagrangian expression for this problem can be

$$L_t = Q_t + \lambda_1 (\sum_{j=2}^{n} w_j \bar{e}_j' - B_\ell) + \lambda_2 (-\sum_{j=2}^{n} w_j \bar{e}_j' + B_u) + \lambda_3 (\sum_{j=1}^{n} w_j - 1) \quad \ldots. (A. 2)$$

where λ_i, $i = 1, 2, 3$, are the Larangian multipliers.
The Kuhn-Tucker conditions are

$$\frac{\partial L_t}{\partial w_1} = \lambda_3 \geqslant 0 \quad \ldots (A.3)$$

$$\frac{\partial L_t}{\partial w_s} = \frac{\partial Q_t}{\partial w_s} + \lambda_1 \bar{e}_s' - \lambda_2 \bar{e}_s' + \lambda_3 \geqslant 0 \quad s=2, \ldots, n \quad \ldots (A.4)$$

$$w_s \geqslant 0 \quad s=1, \ldots, n \quad \ldots (A.5)$$

$$\frac{\partial L_t}{\partial w_s} \cdot w_s = 0 \qquad s=1, \ldots, n \quad \ldots (A.6)$$

$$\sum_{j=1}^{n} w_j \bar{e}_j' - B_\ell \geqslant 0 \text{ and } -\sum_{j=2}^{n} w_j \bar{e}_j' + B_u \geqslant 0 \quad \ldots (A.7)$$

$$\sum_{j=1}^{n} w_j - 1 = 0 \quad \ldots (A.8)$$

$$\lambda_i \geqslant 0, \; i=1, 2, 3 \quad \ldots (A.9)$$

$$\lambda_1 (\sum_{j=2}^{n} w_j \bar{e}_j' - B_\ell) = 0 \text{ and } \lambda_2 (- \sum_{j=2}^{n} w_j \bar{e}_j' + B_u) = 0 \quad (A.10)$$

It is assumed that $w_1 > 0$ alway so that $\lambda_3 = 0$ from the conditions (A.3) and (A.6).
Case 1:

Consider the case when average nominal OWCC falls within the specified bounds. That is, the condition (A. 7) holds strictly and λ_1 and λ_2 become zero. Thus the inequality (A.4) now becomes

$$\frac{\partial L_t}{\partial w_s} = \frac{\partial Q_t}{\partial w_s} + \lambda_3 \geqslant 0, \quad s=2, \ldots, n. \qquad \ldots(A.11)$$

where

$$\frac{\partial Q_t}{\partial w_s} = \sum_{i=1}^{I} a_i \eta^2 (X^i) \{2[w_s - \eta(X^i:e_s)/\eta(X^i)]\ V(e_s')$$
$$+ 2 \sum_{k=1}^{m} \frac{\eta(X^i:Z_k)}{\eta(X^i)} c(e_s'\ Z_k)\}, \quad s=2, \ldots, n \qquad \ldots(A.12)$$

If $\frac{\partial Q_t}{\partial w_s} > 0$, then

$w_s = 0$ from the condition (A.6), and the assumption $w_1 > 0$..
Thus, the sufficient condtion for $w_s = 0$ is obtained by substituting $w_s = 0$ to the equation (A.12), i.e.,

$$\sum_{i=1}^{I} a_i \eta(X^i) \{-\eta (X^i:e_s) V(e_s') + \sum_{k=1}^{m} \eta(X^i:Z_k) c(e_s'\ Z_k)\} > 0,$$
$$\text{for } s \neq 1 \qquad \ldots (A.\ 13)$$

If single currency peg is optimal, then the inequality (A. 13) must hold for all $s \neq 1$[2)]

The interior solution, however, implies all weights are postive, which implies

$$\frac{\partial Q_t}{\partial w_s} = 0, \text{ for } s=2, \ldots n \qquad \ldots(A.14)$$

From the equations (A. 12) and (A. 13) and after some manipulation, one can get an OWCC.

$$w_s = \sum_{i=1}^{I} A_i \left[\frac{\eta(X^i:e_s)}{\eta(X^i)} - \sum_{k=1}^{m} \frac{\eta(X^i:Z_k)}{\eta(X^i)} \cdot \frac{c(e_s'\ Z_k)}{V(e_s')}\right], s=2, \ldots, n \qquad \ldots(A.15)$$

$$\text{where} \quad A_i \equiv \frac{a_i \eta^2 (X^i)}{\sum_{j=1}^{I} a_j \eta^2 (X^j)}, \ i=1, \ldots, I \qquad \ldots(A.16)$$

2) Lipschitz and Sundararajan (1980) discuss extensively on the optimal single currency peg.

and

$$w_1 = 1 - \sum_{j=2}^{n} w_j$$

If authorities concern only a certain target variable, let say X^{ℓ}, then one can obtain the equations (13a) and (13b) in the case of single target by setting $a_{\ell} = 1$ ans $a_i = 0$ for $i \neq \ell$.
That is

$$A_{\ell} = 1 \text{ and } A_i = 0 \text{ for } i \neq \ell.$$

Case 2:

When the average nominal OWCC falls on the boundary, one of the inequalities in (A. 7) holds strictly and the other becomes the equality. First we will examine the case that the lower limit is reached i.e,

$$\sum_{j=2}^{n} w_j \bar{e}_j' - B_{\ell} = 0 \text{ and } -\sum_{j=2}^{n} w_j \bar{e}_j' + B_u > 0. \qquad \text{...(A. 17)}$$

Then we know

$$\lambda_1 > 0 \text{ and } \lambda_2 = 0.$$

so as to satisfy the condition (A.10)
The inequality (A. 4) becomes

$$\sum_{i=1}^{I} a_i \eta^2(X^i) \{[w_s - \frac{\eta(X^i:e_s)}{\eta(X^i)}] V(e_s') + \sum_{k=1}^{m} \frac{\eta(X^i:Z_k)}{\eta(X^i)} c(e_s' Z_k)\} + \frac{\lambda_1}{2} \bar{e}_s' \geqslant 0 \quad s=2, \ldots, n \qquad \text{...(A. 18)}$$

If, however, the inequality (A. 18) holds as a strict inequality, then $w_s = 0$ from the condition (A.6). Following the same procedure as in the case 1, a sufficient condition for $w_s = 0$ is

$$\sum_{i=1}^{I} a_i \eta(X^i) [-\eta(X^i:e_s) \cdot V(e_s') + \sum_{k=1}^{m} \eta(X^i:Z_k) \cdot C(e_s' Z_k)\} + \frac{\lambda_1}{2} \bar{e}_s' > 0, \text{ for } s \neq 1 \qquad \text{...(A.19)}$$

For an interior solution, i.e. $w_s > 0$ for all s, a OWCC can be obtained setting the conditions (A. 17) as an equality,

$$w_s = \sum_{i=1}^{I} A_i [\frac{\eta(X^i:e_s)}{\eta(X^i)} - \sum_{k=1}^{m} \frac{\eta(X^i:Z_k)}{\eta(X^i)} \frac{c(e_s' Z_k)}{V(e_s')}] - \frac{\lambda_1}{2} \frac{1}{A} \cdot \frac{\bar{e}_s'}{V(e_s')},$$

$$s=2, \ldots, n \qquad \text{...(A. 20)}$$

where

$$A \equiv \sum_{j=1}^{I} a_j \eta^2 (X^i)$$

and

$$w_1 = 1 - \sum_{j=2}^{n} w_j$$

λ_1 can be obtained by substituting each w_s to b $B_\ell = \sum_{s=2}^{n} w_s e_s'$ from the condition (A. 17). The explicit expression for λ_1 wil be omitted since the result does not readily yield much economic meaning.

If the upper limit is reached, then the sufficient condition to exclude the s^{th} trading partner from the OWCC will be exactly same as before except we now have $-\lambda_2$ for λ_1 in the condition (A.19). Also the interior solution for the OWCC will be altered by substituting $-\lambda_2$ for λ_1 in the condition (A. 20).

REFERENCES

1. Belanger, Gerard, "An Indicator of Effective Exchange rates for Primary Producing Countries." *Staff Papers, Vol. 23,* March 1976, pp. 113-36.
2. Feltenstein, Andrew, Morris Goldstein, and Susan M. Schadler, "A Multilateral Exchange Rate Model for Primary Producing Countries." *Staff Papers, Vol. 26,* Septembr 1979, pp. 543-82.
3. Lanyi, Anthony and Esther C. Suss, "Exchange Rate Variability: Alternative Measures and Interpretation." *Staff Papers, Vol. 29,* December 1982, pp. 527-560.
4. Lipschitz, Leslie, "Exchange Rate Policy for a Small Developing Countries, and the Selection of an Appropriate Standard," *Staff Papers, Vol. 26,* September 1979, pp. 423-49.
5. Lipschitz, Leslie, and V. Sundararajan, "The Optimal Basket in a World of Generalized Floating," *Staff Papers, Vol. 27,* March 1980, pp. 80-100.
6. Rhomberg, Rudolf., "Indices of Effective Exchange Rates," *Staff Papers, Vol. 23,* March 76, pp. 88-112.
7. Flanders, M. June and Elhanan Helpman, "An Optimal Exchange Rate Peg in a World of General Floating," *Review of Economic Studies,* July 1979.
8. Maciejewski, E. B., "'Real' Effective Exchange Rate Indices, A Re-Examination of the Major Conceptual and Methodological Issues." *Staff Papers, Vol. 30, No. 3,* September 1983, pp. 491-541.