

Erratic Dynamics in an Overlapping Generations Model

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I. Introduction

The problem of stability in an overlapping generations model has recently attracted the attention of the economics profession. Benhabib and Day [1] have shown in a simple, deterministic overlapping generations model that the economy can display very complicated orbit structure similar to one in stochastic economy and generate chaotic dynamics. They also have shown that such chaotic dynamics arise from a very wide class of utility functions. Grandmont [3] has also constructed a deterministic overlapping generations economy and considered situations of Samuelson economy in which the young save and lend to the old. He analyzes the competitive monetary economy and confirms Benhabib and Day's result in that economy undergoes very complicated, deterministic business cycle endogenously. Such endogenous business cycle allows a possibility that the government has the power to stabilize the economy by implementing deterministic counter-cyclical policies.

This paper derives properties of stability when the young generation exhibits impatience and borrows from the old in a two-period classical overlapping generations model. There exist two steady-state equilibria, a no-trade equilibrium and a golden-rule equilibrium. The classical no-trade equilibrium is locally unstable, while the local stability properties of golden-rule equilibrium depends on the relative values of the income and substitution effects of changes in the rate of interest (see Chang et. al. [2]). Benhabib and Day moreover show that there are classes of utility functions such that the trajectories are bounded but do not converge to a cycle of any order. The sufficient substitutability condition of Benhabib and Day are derived from Li-Yorke's chaos theorem. The indifference curves are required to have enough curvatures so that the marginal rates of substitution can vary sufficiently.

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Three important aspects of the Sufficient Substitutability Condition (SSC) of Benhabib and Day's are not yet fully explored. First, it is not clear how the SSC's are derived from within the classical model and what characteristics of the model would be attributed to generating such conditions. Benhabib and Day [1] directly apply Li-Yorke's chaos theorem to find the conditions. Instead, can we derive the Li-Yorke's chaos condition within the model? Second, when can we weaken the SSC? The possible answer may be found in the Grandmont's paper [3] of the different, Samuelson model. The origin of the endogenous chaotic cycle is discussed to be the conflict between the wealth effect and the intertemporal substitution effect with respect to changes in interest rates. Third, even if there were an indication of chaotic behavior, the chaotic set may be of Lebesgue measure zero and thus essentially unobservable. Are nonperiodic fluctuations observable in a world described by the model? We wish to show how the prospects for chaotic behavior with positive measure depend on the parameters of preferences of the young and old, in this way showing how intrinsic dynamics depend on "structural" forces.

Our major results are these. If the so-called arrow-Pratt relative degree of risk aversion is sufficiently higher for young than for old and the degree is nondecreasing, then a chaotic cycle exists when the intertemporal marginal rate of substitution decreases from a value larger than the population growth rate to a value smaller than the population growth rate. i) If the substitution effect exceeds the income effect, there will be no cycle. If the income effect exceeds the substitution effect, there will be no cycle of period larger than one when the old's consumption does not decrease when the interest rate is equal to the population growth rate. ii) If the relative degree of risk aversion is sufficiently higher for young and the degree is non-decreasing, then the substitution effect is surely dominated by the income effect. Finally, the conditions for observable unstable cycle will be given in terms of usual expansivity conditions in measure and ergodic theory.

II. The Model

Following Benhabib and Day [1] we consider dynamic properties of an overlapping generations model and maintain their assumptions and notations (the classical model). A representative individual lives for two periods and when young, determines a (non-negative) consumption for his youth $C_0(t)$ and for high old age $C_1(t+1)$ in order to maximize his utility $U_0(C_0(t)) + U_1(C_1(t+1))$ subject to the budget constraint

$$C_1(t+1) = W_1 + \rho t [W_0 - C_0(t)], \quad C_0(t) \geq 0, \quad C_1(t+1) \geq 0. \quad (1)$$

where ρ , W_0 , W_1 are the interest factor, fixed per-capita endowment of the young and old respectively. If population grows at the rate r , the market-clearing equilibrium condition for the economy as a whole is

$$(1+r) [W_0 - C_0(t)] + W_1 - C_1(t) = 0. \quad (2)$$

Defining $(C_0(\rho_t), C_1(\rho_t))$ to be the consumption vector maximizing the utility of the t^{th} generation subject to (1), the fundamental dynamic equations of Benhabib and Day may be written as:

$$C_1(\rho_t) = W_1 + \rho_t [W_0 - C_0(\rho_t)] \quad (1')$$

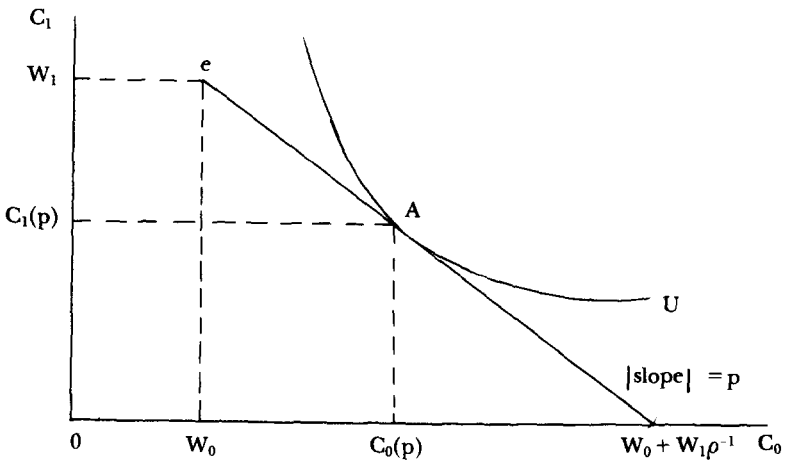
$$(1+r) [W_0 - C_0(\rho_t)] + W_1 - C_1(\rho_{t-1}) = 0 \quad \text{for all } t \geq 0 \quad (2')$$

Since the dynamic properties of the model hinges on the $C_0(\rho_t)$ and $C_1(\rho_t)$ solution functions, let us discuss the characteristics of $C_0(\rho_t)$ and $C_1(\rho_t)$ functions by looking at each individual's choice problem in the classical case. Assuming the utility function is strictly concave, twice differentiable and increasing in its arguments, any interior optimum must satisfy

$$U'_0(C_0(\rho_t)) = \rho_t U'_1(C_1(\rho_t)), \quad \text{or} \quad \frac{U'_0(C_0(\rho_t))}{U'_1(C_1(\rho_t))} = \rho_t \quad (3)$$

in which primes denote partial derivatives.

The intertemporal marginal rate of substitution in consumption is then equal to the interest factor at an interior optimum (See Figure 1).



[Figure 1]

The line $e W_0 + \frac{W_1}{\rho}$ is the budget line, U is the indifference curve, e is the endowment point. Note that to have the classical case, point A must be down to the right of e on the budget line (the young borrow from the old). But this implies that for a given e , the marginal rate of substitution at A must be smaller than the marginal rate of substitution at e . Let $\rho =$

$\frac{U'_0(W_0)}{U'_1(W_1)}$. Then, $\rho \leq \bar{\rho}$ for the classical case. If $\rho \geq \bar{\rho}$, $C_0(\rho) = W_0$, and

$C_1(\rho) = W_1$ so that ρ is the interest factor associated with no-trade equilibrium. If $\rho < \bar{\rho}$, the classical constraint is no longer binding, in which case $C_0(\rho) > W_0$, $0 < C_1(\rho) < W_1$. Also note that in view of (1'), (3) can be rewritten as

$$\begin{aligned} (W_0 - C_0(\rho_t)) U'_0(C_0(\rho_t)) &= (C_1(\rho_t) - W_1) U'_1(C_1(\rho_t)) \text{ for } \rho_t < \bar{\rho} \\ \text{or } (C_0(\rho_t) - W_0) U'_0(C_0(\rho_t)) &= (1 + r) (C_0(\rho_{t+1}) - W_0) \\ U'_1((1 + r) (W_0 - C_0(\rho_{t+1})) + W_1) \end{aligned} \quad (3')$$

Furthermore, $\bar{\rho} > 1 + r$ is assumed in the classical model. (See Gale's Theorem 2 on page 21 [5]).

Such dynamic properties of the model as stability and existence of a cycle depend on whether or not there is an intertemporal substitution effect dominating wealth effect of changes in the interest rates for the young. Therefore, it is necessary to investigate further the properties of $C_0(\rho_t)$ and $C_1(\rho_t)$. Differentiating (3) with respect to ρ_t yields, with time-subscripts dropped,

$$C'_0(\rho) = \frac{U'_1 - \rho U''_1 (C_0 - W_0)}{\Delta} \quad (4)$$

$$C'_1(\rho) = \frac{-U''_0 - (C_0 - W_0) U''_0}{\Delta} \quad (5)$$

where $\Delta \equiv U''_0 + \rho^2 U''_1 < 0$.

Note that $C'_0(\rho) < 0$ for $\rho < \bar{\rho}$. And it is easy to see that $C_0(\rho)$ diverges to infinity as ρ decreases down to zero. (In view of (1') and (3), suppose not. Then C_0 and thus U'_0 are finite and, from (1'), C_1 will be finite as ρ approaches to zero. Thus, from (3), the right-hand side goes to zero while the left-hand side does not, which is a contradiction.) However, the sign of $C'_1(\rho)$ is indeterminate. The first term in the right-hand side of (5) $(-U''_0/\Delta)$ shows the intertemporal substitution effect of changes in the interest rate while the second term $-(C_0 - W_0)U''_0/\Delta$ shows the income (or wealth) effect of the interest rate change. The substitution effect is in a normal direction in the sense that a small increase in the interest rate makes the current consumption of young more expensive and the future consumption of young (when old) less expensive so that he buys less

expensive goods more provided there is no wealth change. But the income effect works in the opposite direction. As the rate of interest goes up, the future consumption demanded decreases due to the real income decrease. If the substitution effect exceeds the income effect, then $C'_1(\rho) > 0$ while if the income effect exceeds the substitution effect $C'_1(\rho) < 0$. To know more about the behavior of $C_1(\rho)$ when ρ is close to $\bar{\rho}$.

First $-\frac{C'_1(\bar{\rho})}{C'_0(\bar{\rho})} = \frac{U'_0(W_0)}{U'_1(W_1)} = \bar{\rho}$, because, at $\rho = \bar{\rho}$, $C_0 = W_0$ and $C_1 = W_1$. Thus $C'_1(\bar{\rho}) > 0$. By continuity, we obtain $C'_1(\rho) > 0$, for ρ close to $\bar{\rho}$ and $\rho < \bar{\rho}$. Next, from (5), $C'_1(\rho) \geq 0$ if and only if $(C_0 - W_0)U''_0 + U'_0 \geq 0$.

Defining the measure of relative risk aversion ($R_0(C_0)$) for consumption for youth as $R_0(C_0) = -U''_0(C_0)C_0/U'_0(C_0)$ and applying it to the above inequalities, we obtain the following:

Fact 1: $C'_1(\rho) \geq 0$ if $R_0(C_0) \leq 1 + \frac{W_0}{C_0 - W_0}$ $\rho < \bar{\rho}$

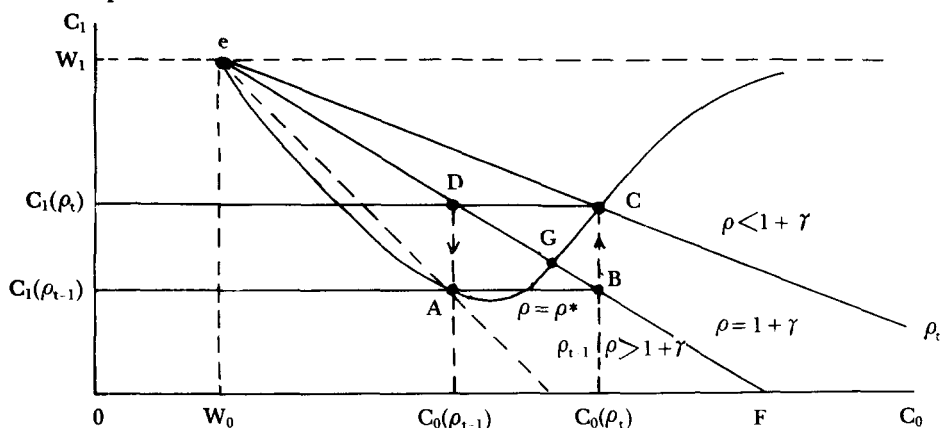
Since $C_0 > W_0$ for $\rho < \bar{\rho}$, the right-hand side of inequality is bounded below by one. Therefore, if $R_0(C_0) \leq 1$ for all $C_0 > W_0$, then $C'_1(\rho) > 0$ for $\rho < \bar{\rho}$. Also if $R_0(C_0)$ is non-decreasing function of C_0 for $C_0 > W_0$ and there is $\tilde{C}_0 > W_0$ such that $R_0(\tilde{C}_0) > 1$,¹⁾ there exists a $\rho^* < \bar{\rho}$ such that $C'_1(\rho^*) = 0$ if $\rho = \rho^*$, $C'_1(\rho) > 0$, if $\rho^* < \rho < \bar{\rho}$ and $C'_1(\rho) < 0$, if $0 < \rho < \rho^*$. This is true by continuity because the right-hand side of the second inequality in Fact 1 monotonically decreases toward one as ρ decreases from $\bar{\rho}$ down toward zero (thus C_0 increases from W_0 to infinity) and R_0 is non-decreasing in C_0 and $R_0(\tilde{C}_0) > 1$ for some \tilde{C}_0 . We summarize these results in the following:

Lemma 1: i) If $R_0(C_0) \leq 1$ for all $C_0 > W_0$, then $C_1(\rho) > 0$ for $\rho < \bar{\rho}$.
ii) If $R_0(C_0)$ is non-decreasing function of C_0 for $C_0 > W_0$ and there is a $\tilde{C}_0 > W_0$ such that $R_0(\tilde{C}_0) > 1$, then there exists a $\rho^* < \bar{\rho}$ such that $C'_1(\rho) < 0$ if $0 < \rho < \rho^*$, $C'_1(\rho^*) = 0$ if $\rho = \rho^*$, and $C'_1(\rho) > 0$ if $\rho^* < \rho < \bar{\rho}$.

Now returning to fundamental dynamic equations (1') and (2'), it is more convenient to write them explicitly in terms of the first-order nonlinear forward difference equation φ in ρ_t . This is feasible because $C_0(\rho_t)$ is invertible as shown before ($C'_0 < 0$) (See the difference between ours and Gale [5] and Benhabib and Day [1]). Thus, we have

$$\begin{aligned} \rho_t &= C_0^{-1} \left(\frac{1}{1+r} (W_1 - C_1(\rho_{t-1})) + W_0 \right) \\ \text{or} \\ &= C_0^{-1} \left(\frac{\rho_{t-1}}{1+r} (C_0(\rho_{t-1}) - W_0) + W_0 \right) \equiv \varphi(\rho_{t-1}). \end{aligned} \quad (6)$$

Clearly, φ is a continuously differentiable mapping from $[0, \bar{\rho}]$ into $[\bar{\rho}, \bar{\rho}]$ in which $\bar{\rho} \equiv C_0^{-1}(\frac{1}{1+\gamma} W_1 + W_0)$. It may be useful to illustrate the forward dynamics in (6) graphically. This is done in Figure 2 below with the assumption in Lemma 1 ii).



[Figure 2]

The line eF is the market-clearing materials balance constraint, $eAGC$ is the offer curve, e is the stationary no-trade equilibrium, G is the stationary (golden-rule) exchange equilibrium. The offer curve is smooth and goes through the endowment point—this corresponds to $\rho \geq \bar{\rho}$. The curve lies below the materials balance constraint when $\rho > 1 + \gamma$, and above when $\rho < 1 + \gamma$. Figure 2 is drawn under the assumption that $\bar{\rho} > 1 + \gamma$ so that the offer curve cuts the materials balance constraint from below. Note that C_0 increases throughout as ρ decreases from $\bar{\rho}$ to 0. Figure 2 also corresponds to the case considered in ii) of Lemma 1, in which $R_0(C_0)$ is non-decreasing and $R_0(\tilde{C}_0) > 1$ for some $\tilde{C}_0 > W_0$. The offer curve has a unique critical point corresponding to the value $\rho = \rho^*$. Starting with ρ_{t-1} such that $\rho > \rho_{t-1} > 1 + \gamma$, we can draw the budget line corresponding to $\rho = \rho_{t-1}$. It intersects the young's offer curve at point A of coordinates $(C_0(\rho_{t-1}), C_1(\rho_{t-1}))$ satisfying (3). Finding the value of ρ_t is then achieved by following the arrows in the figure, by going first horizontally to point B on the materials balance constraint, and then vertically back to the offer curve. This procedure yields the point of coordinate $(C_0(\rho_t), C_1(\rho_t))$ and thus the corresponding value of $\rho_t = \varphi(\rho_{t-1})$ by drawing the corresponding intertemporal budget line p_t . The figure shows that φ has two fixed points, one for $\rho = \bar{\rho}$ (point e) and the other for $\rho = 1 + \gamma$ (point G). Also the figure gives an example of a cycle of period 2 (see the time path of $A B C D$).

So far we have shown that the forward dynamics completely depend on the nonlinear different equation (6). Now, let us summarize the properties of (6) in the following

Lemma 2: φ is continuously differentiable, and

$$\text{i) } \varphi(\bar{\rho}) = \bar{\rho}, \varphi(1+\gamma) = 1+\gamma \begin{cases} \varphi(\rho) < \rho & \text{if and only if } 1+\gamma < \rho < \bar{\rho} \\ \varphi(\rho) > \rho & \text{if and only if } \rho < 1+\gamma \end{cases}$$

$$\text{ii) } \varphi'(\rho_{t-1}) = \frac{-C_1'(\rho_{t-1})}{(1+\gamma)C_0'(\rho_t)} = \frac{C_0(\rho_{t-1}) - W_0 + \rho_{t-1}C_0'(\rho_{t-1})}{(1+\gamma)C_0'(\varphi(\rho_{t-1}))}$$

Thus, $\varphi'(\bar{\rho}) = \frac{\bar{\rho}}{1+\gamma} > 1$, and $\varphi'(1+\gamma) < 1$, ρ is close to

iii) Let $\alpha_0 = \sup R_0(C_0)$. If $\alpha_0 \leq 1$, $\varphi'(\rho) > 0$ for all $\rho \leq \bar{\rho}$.

iv) If $\alpha_0 > 1$ and $R_0(C_0)$ is non-decreasing, there is $\rho^* < \bar{\rho}$ such that $\varphi'(\rho) > 0$ when $\rho^* < \rho \leq \bar{\rho}$, $\varphi'(\rho^*) = 0$, and $\varphi'(\rho) < 0$ when $0 < \rho < \rho^*$. In particular, $\varphi'(1+\gamma) < 0$, $\rho^* > 1+\gamma$, $\rho^* > \varphi(\rho^*)$ are equivalent. Then $\varphi(\rho^*) < 1+\gamma < \rho^*$ and $\varphi(\rho^*) < 1+\gamma < \varphi^2(\rho^*)$.

Proof: Note $\rho_t = \varphi(\rho_{t-1}) = C_0^{-1} \left(\frac{\rho_{t-1}}{1+\gamma} (C_0(\rho_{t-1}) - W_0) + W_0 \right)$

$$= C_0^{-1} \left(\frac{1}{1+\gamma} (W_1 - C_1(\rho_{t-1})) + W_0 \right).$$

i) The first and second part are obvious. The third part is true because by definition of φ , $C_0(\rho_t) - W_0 = \frac{\rho_{t-1}}{1+\gamma} [C_0(\rho_{t-1}) - W_0]$ and $C_0' < 0$

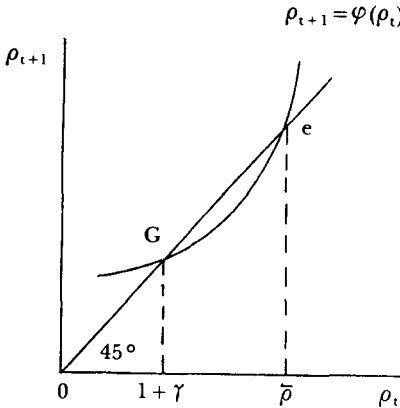
ii) It is straightforward from implicit differentiation of φ and part i) above and $C_0 > W_0$.

iii) It is obvious in view of Lemma 1 i), and part ii) above.

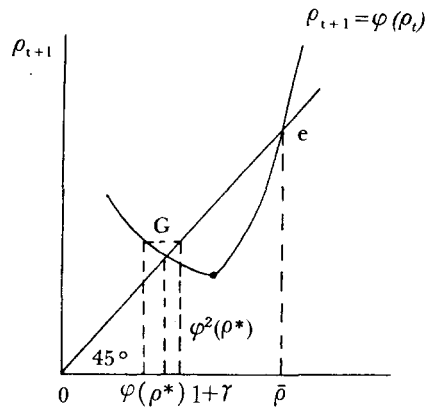
iv) The first part comes easily from Lemma 1 ii), and part ii) above. Since φ has two fixed points ($(1+\gamma)$ and $\bar{\rho}$) and a unique minimum point ρ^* , the second part is true (see the Figure 3.b below). ($\varphi'(1+\gamma) < 0$ implies that ρ^* at which $\varphi'(\rho^*) = 0$ lies larger than $1+\gamma$. The fact that $\rho^* > 1+\gamma$ implies the minimum value of $\varphi(\rho^*)$ must lie below the 45° line, that is $\varphi(\rho^*) < \rho^*$. Finally the fact that the minimum of φ lies below the 45° line and $1+\gamma$ and $\bar{\rho}$ are fixed points implies that $\varphi'(1+\gamma) < 0$, completing the equivalence.) The last part of iv) is obtained by using the fact that $\varphi(\rho^*)$ is the minimum and applying part i) above to the obtained result $\varphi(\rho^*) < 1+\gamma$. (Specifically, $1+\gamma > \varphi(\rho^*)$ because ρ^* is a

minimum point and $1 + \gamma$ is not. Thus, we have $\varphi(\rho^*) < 1 + \gamma < \rho^*$. Applying part i) above to this first inequality yields $\varphi(\rho^*) < \varphi^2(\rho^*)$. But $1 + \gamma < \varphi^2(\rho^*)$ because φ is a decreasing function for any $\rho < 1 + \gamma$ due to $\varphi'(1 + \gamma) < 0$ and one of such ρ is $\varphi(\rho^*)$ and therefore $\varphi^2(\rho^*) = \varphi(\varphi(\rho^*)) > \varphi(1 + \gamma) = 1 + \gamma$. This ends the proofs of Lemma 2.

The dynamics of the model may be illustrated graphically in Figure 3.



[Figure 3-a]



[Figure 3-b]

Figure 3-a is the graphical counterpart to Lemma 2 iii), and Figure 3-b is the graphical counterpart to Lemma 2 iv) when $\rho^* > 1 + \gamma$. Both of these cases show that the no-trade equilibrium e is unstable in the classical model. The (golden-rule) trade equilibrium is stable in Figure 3-a while it is not in Figure 3-b. Stability essentially depends on the income and substitution effect of a change in interest rate (see Chang et. al. [2]). If the substitution effect for C_1 exceeds the income effect ($C'_1(\rho) > 0$), then the dynamics are displayed as in Figure 3-a. If the income effect exceeds the substitution effect for certain values of ρ , then Figure 3-b type dynamics are displayed.

We will now characterize the dynamics of φ in terms of the utility function, following Benhabib and Day. Indeed, the dynamics of φ are essentially contained in (3'):

$$(C_0(\rho_t) - W_0) U'_0(C_0(\rho_t)) = (1 + \gamma) (C_0(\rho_{t-1}) - W_0) U'_1((1 + \gamma) (W - C_0(\rho_{t+1}) + W_1)), \text{ for } \rho_t \leq \bar{\rho}. \quad (3')$$

Defining $C_t \equiv C_0(\rho_t)$, $V_1(C_t) \equiv (C_t - W_0) U'_0(C_t)$ for $C_t \in [W'_0, \infty)$ and

$$V_2(C_t) \equiv (1 + \gamma) (C_t - W_0) U'_1((1 + \gamma) (W_0 - C_t) + W_1), \text{ for } C_t \in [W_0, W_0 + \frac{W_1}{1 + \gamma}]$$

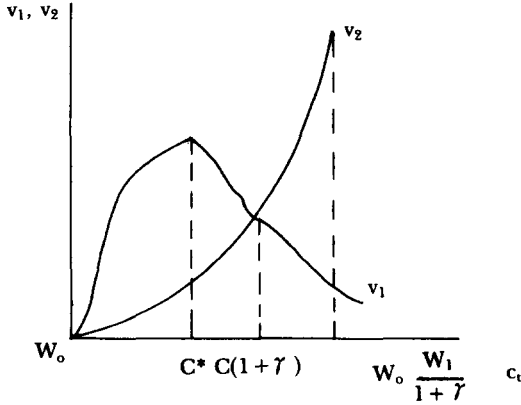
we may rewrite (3') as :

$$V_1(C_t) = V_2(C_{t+1}) \quad (7)$$

Note that there is a one-to-one correspondence between C_t and ρ_t and $dC_t/d\rho_t < 0$. It is easily shown that when $\alpha_o > 1$ and R_o is non-decreasing, $v_1'(c_t) = U'_o + (C_t - W_o)U''_o > 0$ if $\rho^* < \rho \leq \bar{\rho}$, ($w_o \leq C_t < C^*$), $v_1' = 0$ if $\rho = \rho^*$ ($C_t = C^*$), and $v_1' < 0$ if $0 < \rho < \rho^*$ ($C_t > C^*$), in which $C^* = C'_o(\rho^*)$. Also, it is obvious that $v_2'(C_t) = (1 + r)U'_1 - (1 + r)^2(C_t - W_o)U''_1 > 0$ for $C_t \in [W_o, W_o + \frac{W_1}{1+r}]$.

Therefore v_2 is an increasing function of C_t from $[W_o, W_o + \frac{W_1}{1+r}]$ into $[0, \infty)$ and v_1 maps $[W_o, \infty)$ into $[0, \infty)$. (See Figure 4 below). Then the explicit relationship between C_t and C_{t+1} exists if we define the map x as

$$C_{t+1} = x(C_t) \equiv (v_2^{-1} \circ v_1)(C_t) \text{ for all } C_t \in [W_o, \infty). \quad (8)$$



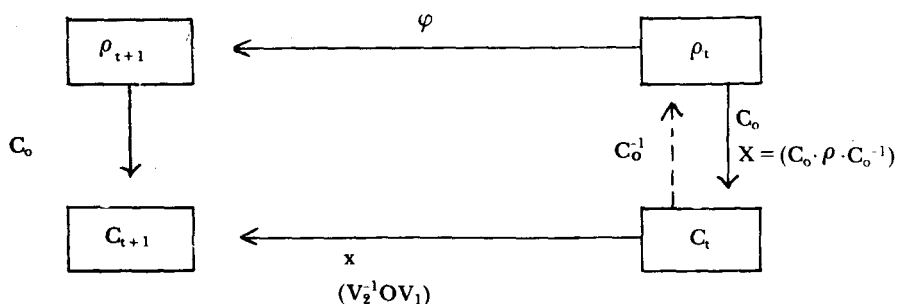
($\alpha_o > 1$, R_o is non-decreasing)

[Figure 4]

Note that x is well-defined and maps $[W_o, \infty)$ into $[W_o, W_o + \frac{W_1}{1+r}]$. Also note that, if $\alpha_o > 1$ R_o is non-decreasing, v_1 has a unique maximum at $C_t = C^*$, and that if $\alpha_o \leq 1$, v_1 is increasing everywhere (see Figure 4). Since the utility function is strictly concave, v_1 and v_2 are bounded as follows.

- Fact 2:** i) $V_1(C_t) < (C_t - W_o) U'_o(W_o)$, and
ii) $V_2(C_t) > (1 + r)(C_t - W_o) U'_1(W_1)$.

The properties of x can be found in Lemma 2 above because of the one-to-one correspondence between C_t and ρ_t (see Figure 5 below).



[Figure 5]

For example, $x' = (C'_0 \cdot \rho' \cdot (C_0^{-1})')$. In the case of $C_t = W_0$, then

$$x'(W_0) = \frac{U_0(W_0)}{(1+r)U'_1(W_0)} = \frac{\bar{\rho}}{1+r} = \varphi'(\bar{\rho}) \quad \text{as expected. (Note that}$$

$$x'(C_t) = \frac{V'_1(C_t)}{V'_2(C_{t+1})}. \quad \text{One property of } x \text{ which will be useful later and}$$

not shown in Lemma 2 is given in the following.

Lemma 3: Let $C_{t+1} = x(C_t) = v_2^{-1} \circ v_1(C_t)$. Then $x(t) < \frac{\bar{\rho}}{1+r} (C_t - W_0) + W_0$ for all $C_t \in [W_0, \infty)$.

Proof: From Fact 2 ii), it is obvious that $v_2^{-1}((1+r)(\bar{C}_t - W_0)U'_1(W_1)) < \bar{C}_t$ for any $\bar{C}_t \in [W_0, W_0 + \frac{W_1}{1+r})$ because v_2 is increasing.

Choose \bar{C}_t such that $v_1(C_t) = (1+r)(\bar{C}_t - W_0)U'_1(W_1)$ for $C_t \in [W_0, W_0 + \frac{W_1}{1+r})$ namely $\bar{C}_t = \frac{v_1(C_t)}{(1+r)U'_1(W_1)} + W_0$ (such \bar{C}_t is in $[W_0, W_0 + \frac{W_1}{1+r})$)

because of (1')). Therefore for such \bar{C}_t , $x(C_t) = V_2^{-1}(V_1(C_t))$

$$< \frac{v_1(C^*)}{(1+r)U'_1(W_1)} + W_0 < \frac{(C_t - W_0)U'_0(W_0)}{(1+r)U'_1(W_1)} + W_0 = \frac{\bar{\rho}}{1+r} (C_t - W_0) + W_0.$$

The last inequality is true in the view of Fact 2 i) above. This completes the proof of the lemma. Therefore, each period young's consumption is bounded above by the youth endowment plus $\frac{\bar{\rho}}{1+r} (> 1)$ times the difference between the previous period's consumption of young and the endowment. Now, we are in a position to discuss existence of periodic equilibria.

III. Existence of Periodic Equilibria

Three distinct cases of periodic equilibria are now in order; a cycle of period one (global stability), a cycle of period two, and a cycle of period three. Then following Li and Yorke's theorem, a cycle of period three implies cycles of all orders. We first state:

Lemma 4: Given a classical model, either if $\alpha_0 \leq 1$, or if $\alpha_0 > 1$, R_0 is non-decreasing and $C_1' (1 + \tau) \geq 0$, then the map φ has no cycle with period $k \geq 2$. Moreover, the unique stationary (golden-rule) exchange equilibrium is globally φ -stable, i.e., $\lim_{k \rightarrow \infty} \varphi^k(\rho) = 1 + \tau$ for every $\rho < \bar{\rho}$. Note: $C_1' (1 + \tau) \geq 0$, $\rho^* \leq 1 + \tau$ and $\varphi' (1 + \tau) \geq 0$ are all equivalent (see Lemma 2 iv)).

Proof: As shown in Lemma 2, either i) ϕ is increasing everywhere or ii) φ has a unique minimum point $\rho^* < \rho$. Now let $\rho_0 \neq \rho$ or $1 + \tau$.

- i) If $\rho_0 < 1 + \tau$, then $\rho_0 < \varphi_0(\rho_0) < \varphi(1 + \tau) = 1 + \tau$ using Lemma 2 i) and iii). Applying the same arguments to $\varphi^j(\rho_0)$, we get $\varphi^j(\rho_0)$ converges monotonically to $1 + \tau$ when j increases to infinity. Similarly we obtain the reverse inequalities whenever $\rho_0 > 1 + \tau$ and the same convergence result.
- ii) Recall that $\varphi(\rho) \geq \varphi(\rho^*)$ for all ρ . Therefore, without loss of generality, we may assume $\rho_0 \in [\varphi(\rho^*), \rho]$ (If $\rho_0 \in [\varphi(\rho^*), \bar{\rho})$, then choose new $\rho_0 = \varphi(\rho_0) (\geq \varphi(\rho^*))$ Since $C_1' (1 + \tau) \geq 0$, $\varphi' (1 + \tau) \geq 0$ and $\rho^* \leq \varphi(\rho^*)$ applying the same argument as in Lemma 2 iv). Since $\varphi(\rho^*) \geq \rho^*$ $\varphi'(\rho) > 0$ for all $\rho \in [\varphi(\rho^*), \bar{\rho})$. Now applying the argument in i), we obtain the required result. (i.e., if $\rho_0 < 1 + \tau$, then $\rho_0 < \varphi(\rho_0) < \varphi(1 + \tau) = 1 + \tau$ etc.).

There the conditions for a cycle of period $k \geq 2$ are that a) young are sufficiently risk averse ($\alpha_0 > 1$), b) their relative risk aversion (R_0) is non-decreasing, and c) $C_1' (1 + \tau) < 0$. (If we assume b), then a) and c) are the necessary conditions). The last condition is equivalent to $\varphi' (1 + \tau) < 0$ or $\rho^* > 1 + \tau$. The substitution effect for C_1 must be dominated by the income effect when $\rho = 1 + \tau$. Or equivalently, $v_1(C^*) > v_2(C^*)$ ($x(C^*) > C^*$) in which $C^* \equiv C_0(\rho^*)$ is recalled (see Figure 4).

Remark: It is interesting to note that Chang-Kemp-Long [2] proved that the golden-rule equilibrium in the classical model is *locally* stable if and only if the marginal propensity to consume of young exceeds some critical number. It can be easily shown that this requirement is met with a

use of our one of conditions of global stability: the substitution effect dominate income effect when $\rho = 1 + \gamma$ ($C_1' (1 + \gamma) \geq 0$) (If $\alpha_0 \leq 1$, of course $C_1' (1 + \gamma) > 0$). Under this condition, $(C_0 - W_0) U_0'' + U_0' = 0$ when $p = 1 + \gamma$. Chang-Kemp-Long's local stability condition for additive utility function can be written as $-1/2 (1 + \gamma)^2 U_1'' + 1/2 U_0'' + \frac{(1 + \gamma) U_1'}{C_0 - W_0} > 0$ (See (8d) of Chang et. al. [2]). It is easy to check that the latter inequality is implied by our inequality

$$(C_0 - W_0) U_0'' + U_0' \geq 0 : -1/2(1 + \gamma)^2 U_1'' + 1/2 U_0'' + \frac{(1 + \gamma) U_1'}{C_0 - W_0} > -1/2(1 + \gamma)^2 U_1'' - 1/2 \frac{U_0'}{C_0 - W_0} + \frac{U_0'}{C_0 - W_0} = -1/2(1 + \gamma)^2 U_1'' + 1/2 \frac{U_0'}{C_0 - W_0} > 0.$$

Now let us turn to a cycle of period 2. From now on we assume that young's relative risk aversion is nondecreasing. Thus, the necessary conditions for any cycle of period $k \geq 2$ are a) $\alpha_0 > 1$ and c) $C_1' (1 + \gamma) < 0$. In addition, in order to have a cycle of period two, φ^2 must have a fixed point different from $1 + \gamma$ and ρ . Note that φ^2 maps $(0, \rho]$ into (p, p) and continuously differentiable. Also $\varphi^2(\rho) = \rho$, $\varphi^2(\bar{\rho}) = (\frac{\bar{\rho}^2}{1 + \gamma}) > 1$ and $\varphi^2(1 + \gamma) = 1 + \gamma$, using Lemma 2. Therefore, a sufficient condition for φ^2 having a fixed point different from $1 + \gamma$ and $\bar{\rho}$ is $D\varphi^2(1 + \gamma) = \varphi'(1 + \gamma)^2 > 1$. Or, since $\varphi'(1 + \gamma) < 1$ from Lemma 2 ii), $D\varphi^2(1 + \gamma) > 1$ is equivalent to

$$\varphi'(1 + \gamma) < -1. \text{ Using Lemma 2 ii), } \varphi'(1 + \gamma) = \frac{-C_1'(1 + \gamma)}{(1 + \gamma)C_0'(1 + \gamma)} < -1 \text{ or}$$

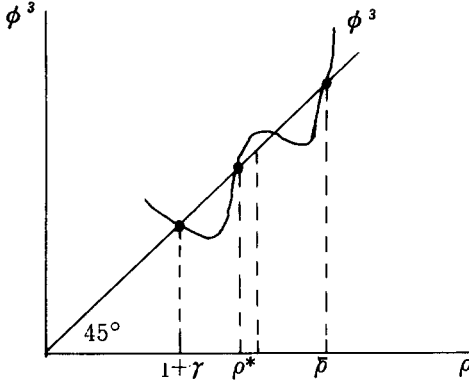
$C_1' (1 + \gamma) < (1 + \gamma) C_0' (1 + \gamma)$, which upon manipulation becomes

$$R_0 > \frac{-2C_0}{C_0 - W_0} + \frac{(1 + \gamma) R_1 C_0}{C_1} \text{ when } \rho = 1 + \gamma, \quad (9)$$

in which R_1 is the old's relative degree of risk aversion. Therefore, a cycle of period two appears if young are sufficiently more risk averse than old.

Now, we are ready to discuss a cycle of period three. (According to Li and Yorke, if a map of an interval to itself has a cycle of period three, then there is an uncountable set of initial conditions which give rise to chaotic trajectories (which may or may not have Lebesgue measure zero).) An orbit $(\rho_1^*, \rho_2^*, \rho_3^*)$ has a cycle of period three if each ρ_1^* is a fixed point of φ^3 , $\rho_1^* \neq \varphi(\rho_1^*) = \rho_2^*$ and $\rho_2^* \neq \varphi^2(\rho_1^*) \equiv \rho_3^* \neq \rho_1^*$. Note that φ^3 maps $(0, \bar{\rho}]$

into $(\bar{\varphi}, \varphi]$ and is continuously differentiable. Also $\varphi^3(\bar{\rho}) = \bar{\rho}$, $D\varphi^3(\bar{\rho}) = (\frac{\bar{\rho}}{1+\gamma})^3 > 1$ and $\varphi^3(1+\gamma) = 1+\gamma$. A sufficient condition under which a cycle of period three exists is that there is ρ^* such that $\rho^* > 1+\gamma$, and $\varphi^3(\rho^*) \geq \rho^*$. The reason why the condition is sufficient may be seen from inspection of graph of φ^3 in Figure 6 below.



[Figure 6]

Note that it is impossible to have $D\varphi^3(1+\gamma) > 1$, because $\varphi'(1+\gamma) < 1$, and $D\varphi^3(1+\gamma) = [\varphi'(1+\gamma)]^3$. Thus, all we need is to make sure that $\varphi^3 \geq \rho$ for some ρ between $1+\gamma$ and $\bar{\rho}$, and we confine such ρ to be ρ^* (the minimum point of φ). Summarizing these results, we have

Proposition 1: Assume that R_0 is nondecreasing. If $\alpha_0 > 1$, $C_1'(1+\gamma) < 0$ (or equivalently, $\rho^* > 1+\gamma$ or $\varphi'(1+\gamma) < 0$), and $\varphi^3(\rho^*) \geq \rho^*$, then the map φ has a cycle of period three. Furthermore, the following string of inequalities holds:

$$\varphi(\rho^*) < 1+\gamma < \rho^* \leq \varphi^3(\rho^*) < \varphi^2(\rho^*). \quad (10)$$

Proof: The first part follows from discussion prior to the statement of Proposition 1 and using continuity of φ^3 map. To prove (10), we need to show only the last inequality in view of Lemma 2 iv). Using the last part of Lemma 2 iv) again, $1+\gamma < \varphi^2(\rho^*)$, to which we apply Lemma 2 i). Then $\varphi^3(\rho^*) < \varphi^2(\rho^*)$, as required. This ends the proof of the proposition.

Now let us interpret these results in terms of utility functions. First, we need the following:

Remark: Here $\rho^* > 1 + r$ is equivalent to $C^* (= C_0(\rho^*)) < C_0(1+r)$ or $C^* < x(C^*)$. And $\varphi^3(\rho^*) \geq \rho^*$ is equivalent to $x^3(C^*) \leq C^*$. (These are due to $C_0' < 0$ for all $\rho < \bar{\rho}$). Therefore, let us write down a x -version of Proposition 1 for record, which is obviously.

Proposition 1': Assume R_0 is nondecreasing. If $\alpha_0 > 1$, $C^* < x(C^*)$ (or equivalently, $v_1(C^*) > v_2(C^*)$), and $x^3(C^*) \leq C^*$, then x has a cycle of period three. Furthermore, the following holds:

$$x^2(C^*) < x^3(C^*) \leq C^* < C_0(1+r) < x(C^*). \quad (10')$$

Indeed, the condition $x^3(C^*) \leq C^*$ can be rewritten in terms of restriction on utility functions:

Lemma 5: Suppose $C^* < x(C^*)$. i If there is a \bar{C} such that $C^* \leq \bar{C} < x(C^*)$ and $v_1(\bar{C}) v_2((1+r)\bar{\rho}^{-1}(C^*-W_0)+W_0)$ or $V_1(\bar{C}) \leq (1+r)^2 \bar{\rho}^{-1}(C^*-W_0) U'_1(W_1)$, then $x^3(C^*) < C^*$.

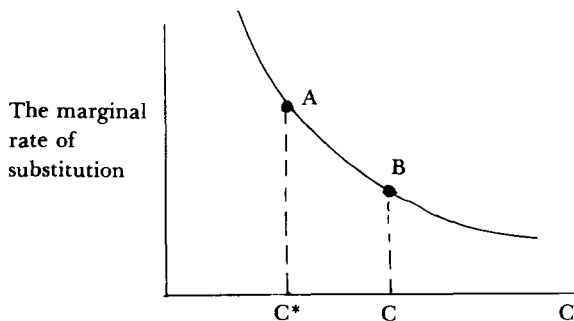
Proof: For such \bar{C} as $C^* < \bar{C} < x(C^*)$, $x^2(C^*) \leq x(\bar{C})$ since $X'(C) \leq 0$ for $C > C^*$. Using Lemma 3 together with the above result, we obtain:

$$x^3(C^*) < \bar{\rho}(1+r)^{-1}(x^2(C^*)-W_0)+W_0 < \bar{\rho}(1+r)^{-1}(x(\bar{C})-W_0)+W_0.$$

But the last term is less than C^* in view of $V_1(\bar{C}) \leq V_2((1+r)\bar{\rho}^{-1}(C^*-W_0)+W_0)$ (Note that Fact 2 ii) makes the condition in the parenthesis in Lemma 5 even stronger). Thus the proof is complete.

Therefore, we can now summarize the results corresponding to Benhabib and Day's chaos Theorem 1 [1]:

Theorem 1: Suppose R_0 is nondecreasing. If $\alpha_0 = \sup R_0 > 1$, and there is a $\bar{C} \geq C^*$ such that (i) $V_1(C^*) > V_2(\bar{C})$ and (ii) $V_1(\bar{C}) \leq V_2((1+r)\bar{\rho}^{-1}(C^*-W_0)+W_0)$ (or $V_1(\bar{C}) \leq (1+r)^2 \bar{\rho}^{-1}(C^*-W_0) U'_1(W_1)$) then the dynamic system has a cycle of period three. Also (10') holds.



[Figure 7]

Proof: It is obvious from Lemma 5 and Proposition 1'. (Note $v_2(\bar{C}) \geq v_2(C^*)$ since v_2 is increasing).

The sufficient conditions for a cycle of period three in Theorem 1 can be interpreted in terms of the Sufficient Substitutability Conditions in Benhabib and Day. First it is easily seen that condition (i) $v_1(C^*) > v_2(C^*)$ is equivalent to $\frac{U'_0(C^*)}{U'_1(C^*)} > 1+\gamma$ (See point A in Figure 7). The marginal rate of substitution evaluated at C^* , the unique maximum point of v_1 , must be larger than $1+\gamma$. Note that this corresponds to one of Benhabib and Day's Sufficient Substitutability Condition ((i) or (ii) of theirs). Second, condition (ii) here requires that for some point $\bar{C} > C^*$ the marginal rate of substitution at $C = \bar{C}$ be less than or equal to a value less than a fraction of (See point B in Figure 7) $1+\gamma$. To see it, divide condition (ii) by $v_2(\bar{C})$. Then we

$$\text{obtain } \frac{v_1(\bar{C})}{v_2(\bar{C})} = \frac{1}{1+\gamma} \frac{U'_0(\bar{C})}{U'_1(\bar{C})} \leq \frac{v_2(\bar{C})}{v_2(\bar{C})}, \text{ in which } \bar{C} \equiv (1+\gamma)\rho^{-1}(C^*-W_0)+W_0 < C^*.$$

Thus, the marginal rate of substitution evaluated at $C = \bar{C}$ must be less than $1+\gamma$ times $\frac{V_2(\bar{C})}{V_2(\bar{C})}$ (< 1 , because v_2 is increasing). Again, condition (ii) corresponds to one of Benhabib and Day's Sufficient Substitutability Condition ((iii) of theirs, for instance). Third, in comparing (10)' with Benhabib and Day's Condition, the first inequality of (10)' (or one of (i) or (ii) of Benhabib and Day's) has not been assumed, instead has been derived within the model. Then it seems that our conditions have fewer requirements than the original Benhabib and Day's. However, we have imposed additional restrictions on utility functions; R_0 be non-decreasing and $\alpha_0 > 1$. Also, we have confined ourselves only to additively separable utility functions.

In the economy of nondecreasing relative risk aversion utilities of the young, which is quite plausible, nevertheless, a) $\alpha_0 > 1$ and b) $v_1(C^*) > v_2(C^*)$ (or (i) in the theorem) are the necessary conditions for existence of a cycle of period three (Lemma 4). Therefore, we may cautiously conclude that our sufficient condition (ii) *weakens* Benhabib and Day's Sufficient Substitutability Condition under these circumstances (additive utility plus nondecreasing R_0).

It is also easily shown that the utility functions satisfying (i) and (ii) of Theorem 1 form a very large class of functions. To construct such functions, choose $C^* < \bar{C} < W_0 + \frac{W_1}{1+\gamma}$ and determine the maximum value

$v_1(C^*)$. Then choose $U'_0(C^*)$ so high that (i) holds. Finally, choose $U'_0(\bar{C}) < U'_0(W_0)$ so low that (ii) holds. Note that we have fixed all the characteristics of the model, except the young's utility function (U_0).

IV. Conclusion

We have identified the main mechanism through which endogenous cycles emerge in the classical overlapping generation's model. The main mechanism is the conflict between the intertemporal substitution and income effects which are associated with a variation of the real rate of interest of borrowing by the young from the old generation. A worthwhile extension of the current work is an inclusion of the various features of the nonseparable preference and the production technology which are nonlinear.

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